Real Analysis Notes

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Chapter 1 Metric Spaces (MTH 511)

1.1 Metric Spaces and Normed Vector Spaces

1.1.1 Metrics

Definition 1.1

Let M be any set. A function $d : M \times M \to [0, \infty)$ is a **metric** on M if it satisfies the following:

1. $0 \leq d(x, y) < \infty, \forall x, y \in M$
2. $d(x, y) = d(y, x), \forall x, y \in M$
3. $d(x, y) = 0 \iff x = y$
4. $d(x, y) \leq d(x, z) + d(z, y)$

1.1.2 Discrete Metric

Example 1.1

The **discrete metric** is defined by:

$$d(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases} \tag{1.1}$$

1.1.3 Norms

Definition 1.2

Let V be a vector space. A **norm** on V is a function $\|\cdot\| : V \to [0, \infty)$ satisfying the following properties:

1. $0 \leq \|\mathbf{x}\| < \infty, \forall \mathbf{x} \in V$
2. $\|\mathbf{x}\| = 0 \iff \mathbf{x} = \mathbf{0}$
3. $\|\alpha \mathbf{x}\| = |\alpha| \|\mathbf{x}\|$
4. $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$

1.1.4 Common Norms

Example 1.2

$$\|\mathbf{x}\|_1 = \sum_{i=1}^{N} |x_i| \tag{1.2}$$

$$\|\mathbf{x}\|_2 = \left(\sum_{i=1}^{N} |x_i|^2 \right)^{1/2} \tag{1.3}$$

$$\|\mathbf{x}\|_p = \Big(\sum_{i=1}^{N} |x_i|^p\Big)^{1/p} \tag{1.4}$$

$$\|\mathbf{x}\|_\infty = \max_{1\le i\le N}(|x_i|) \tag{1.5}$$

1.1.5 Norms of Continuous Functions

Definition 1.3

Consider the set of all continuous function on $[a, b]$. The following are norms on $C([a, b])$.

$$\|f\|_1 = \int_a^b |f(t)|dt \tag{1.6}$$

$$\|f\|_2 = \Big(\int_a^b |f(t)|^2 dt\Big)^{1/2} \tag{1.7}$$

$$\|f\|_p = \Big(\int_a^b |f(t)|^p dt\Big)^{1/p} \tag{1.8}$$

$$\|f\|_\infty = \sup_{t\in[a,b]}(|f(t)|) \tag{1.9}$$

1.1.6 ℓ_p Spaces

Definition 1.4

For p satisfying $1 \le p < \infty$, ℓ_p is the set of all sequences of real numbers $x = (x_i)_{i\in\mathbb{N}}$ for which the following is true:

$$\sum_{i=1}^{\infty} |x_i|^p < \infty \tag{1.10}$$

Definition 1.5

ℓ_∞ is the set of all bounded sequences of reals.

$$\begin{aligned} \|x\|_\infty = \sup_{i\in\mathbb{N}}\{|x_i|\} < c \\ \text{for some } c > 0 \end{aligned} \tag{1.11}$$

☞ **Remark**

$\ell_q \subseteq \ell_p, \forall q \le p$

Theorem 1.1 Hölder's Inequality

Let $p \in (1, \infty)$ and let q satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Given $x \in \ell_p$ and $y \in \ell_q$, we have the following inequality:

$$\sum_{i=1}^{\infty} |x_i y_i| = \|xy\|_{\ell_1} \leq \|x\|_{\ell_p} \|y\|_{\ell_q} \tag{1.12}$$

1.1.7 Metric Spaces

Definition 1.6

The set M, equipped with the metric d defines a metric space (M, d).

Definition 1.7

Given $x \in (M, d)$ and $r > 0$, the **Open Ball** of radius r centered at x is defined by

$$B_r(x) = \{y \in M \mid d(x, y) < r\} \tag{1.13}$$

Definition 1.8

$A \subseteq M$ is **bounded** if and only if given any $x \in M$, $\exists r > 0$ such that $A \subseteq B_r(x)$.

Definition 1.9

The **diameter** of A is defined as

$$\operatorname{diam}(A) = \sup\{d(x, y) \mid x, y \in A\} \tag{1.14}$$

Definition 1.10

A **neighborhood** of $x \in M$ is any set containing an open ball centered at x.

1.1.8 Convergent and Cauchy Sequences

Definition 1.11 Convergence

A sequence $(x_n) \in M$ converges to $x \in M$ if $d(x_n, x) \to x$ as $n \to \infty$.

Definition 1.12 Convergence

A sequence $(x_n) \in M$ converges to $x \in M$ if, given some $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$ we have $d(x_n, x) < \epsilon$.

Definition 1.13 Convergence

A sequence $(x_n) \in M$ converges to $x \in M$ if, given some $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\{x_n \mid n \geq N\} \subseteq B_\epsilon(x)$.

Definition 1.14 Cauchy

A sequence (x_n) is **Cauchy** if, given some $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall m, n \geq N$ we have $d(x_m, x_n) < \epsilon$.

☞ **Remark**
Every convergent sequence in (M, d) is Cauchy.

☞ **Remark**
Any Cauchy sequence with a convergent subsequence in (M, d) converges in (M, d).

1.2 Topology of Metric Spaces

1.2.1 DeMorgan's Laws

Definition 1.15

$$\Big(\bigcap_{i\in\mathbb{I}} A_i\Big)^c = \bigcup_{i\in\mathbb{I}} A_i^c \tag{1.15}$$

$$\Big(\bigcup_{i\in\mathbb{I}} A_i\Big)^c = \bigcap_{i\in\mathbb{I}} A_i^c \tag{1.16}$$

1.2.2 Limit Points

Definition 1.16

Let A be a subset of (M, d). $x \in M$ is a **limit point** of A if

$$\big(B_\epsilon(x) - \{x\}\big) \cap A \neq \varnothing \tag{1.17}$$

for all $\epsilon > 0$.

Definition 1.17

Let A be a subset of (M, d). $x \in M$ is an **isolated point** of A if

$$(B_\epsilon(x) - \{x\}) \cap A = \varnothing \tag{1.18}$$

If x is not a limit point, it is an isolated point (and vice versa).

Definition 1.18 Boundary Points

Let A be a subset of M. $x \in M$ is a **boundary point** of A if and only if

$$\begin{aligned} (B_\epsilon(x) - \{x\}) \cap A &\neq \varnothing \\ \text{and} \\ (B_\epsilon(x) - \{x\}) \cap A^c &\neq \varnothing \end{aligned} \tag{1.19}$$

1.2.3 Open Sets

Definition 1.19

A set $U \subseteq (M, d)$ is **open** if $\forall x \in U, \exists \epsilon > 0$ such that $B_\epsilon(x) \subset U$.

☞ **Remark**

$\forall x \in M$ and $\forall \epsilon > 0$, $B_\epsilon(x)$ is an open set.

Theorem 1.2

An arbitrary union of open sets is open.

$$V = \bigcup_{\alpha \in A} U_\alpha \text{ is open.} \tag{1.20}$$

Theorem 1.3

A finite intersection of open sets is open.

$$V = \bigcap_{i=1}^{N} U_\alpha \text{ is open.} \tag{1.21}$$

Theorem 1.4

If U is open and $U \subset \mathbb{R}$, then U is a countable union of disjoint, open intervals.

$$\begin{aligned} U &= \bigcap_{n=1}^{\infty} I_n \\ I_n &= (a_n, b_n) \\ I_n \cap I_m &= \varnothing \\ n &\neq m \end{aligned} \tag{1.22}$$

Theorem 1.5

A set U is open if and only if, whenever $(x_n) \in M \to x \in U$, for all but finitely many n, $x_n \in U$.

Definition 1.20

let (U_α) be the set of all open sets in M. (U_α) is an **open base** for M if

$$M = \bigcup (U_\alpha) \tag{1.23}$$

1.2.4 Closed Sets

Definition 1.21

A set $F \subseteq (M, d)$ is closed if and only if $F^c = M - F$ is open.

Definition 1.22

A set $F \subseteq (M, d)$ is closed if and only if, given $x \in M$, $\forall \epsilon > 0$,

$$B_\epsilon(x) \cap F \neq \varnothing \Rightarrow x \in F \tag{1.24}$$

Definition 1.23

A set $F \subseteq (M, d)$ is closed if and only if, given a sequence $(x_n) \subseteq F$

$$(x_n) \to x \in M \Rightarrow x \in F. \tag{1.25}$$

In other words, F is closed if it contains all its limit points.

Definition 1.24 Interior

The **interior** of A is defined as

$$\text{int}(A) = A^\circ = \{x \in A \mid B_\epsilon(x) \subset A \text{ for some } \epsilon > 0\} \tag{1.26}$$

Definition 1.25 Closure

The **closure** of A is defined as

$$\text{cl}(A) = \overline{A} = \bigcap \{F \mid F \text{ is closed and } A \subseteq F\} \tag{1.27}$$

Theorem 1.6

$x \in \overline{A} \iff B_\epsilon(x) \cap A \neq \varnothing, \forall \epsilon > 0.$

Theorem 1.7

$x \in \overline{A} \iff \exists (x_n) \subset A \text{ with } (x_n) \to x.$

1.2.5 Relative Metrics

☞ **Remark** Notation

For $x \in A$ with $A \subseteq M$:

$$B_\epsilon^A(x) = \{y \in A \mid d(x, y) < \epsilon\} = A \cap \{y \in M \mid d(x, y) < \epsilon\} = A \cap B_\epsilon^M(x) \tag{1.28}$$

Definition 1.26

A subset $G \subseteq A$ is open relative to A if, given $x \in G$, $\exists \epsilon > 0$ such that

$$B_\epsilon^A(x) = A \cap B_\epsilon^M(x) \subseteq G \tag{1.29}$$

Corollary 1.1

A subset $G \subseteq A$ is open relative to A if and only if

$$A = G \cap U \tag{1.30}$$

for some U open in A.

Definition 1.27

A set $F \subseteq A$ is closed relative to A if $F^c = A - F$ is open in A.

Corollary 1.2

A subset $F \subseteq A$ is closed relative to A if and only if

$$F = A \cap V \tag{1.31}$$

for some V closed in A.

1.2.6 Seperable Sets

Definition 1.28

A subset of a metric space, $D \subseteq M$, is **dense** in M if it satisfies any of the following:

1. $x \in M \Rightarrow x \in D'$
2. $\forall x \in M$ and $\forall \epsilon > 0$, $B_\epsilon(x) \cap D \neq \varnothing$
3. $U \cap D \neq \varnothing$ for all non-empty U in M
4. $(D^c)^\circ = \varnothing$

Definition 1.29

A set D is **countable** if there exists

$$f : D \to \mathbb{N},\ f \text{ is injective.} \tag{1.32}$$

Definition 1.30

A subset of a metric space, $D \subseteq M$, is **seperable** if it is countable and dense in M.

1.3 Continuity

1.3.1 Continuous Functions

Definition 1.31

Let $f : (M, d) \to (N, \rho)$. f is **continuous** at $x \in M$ if, given $\epsilon > 0$, $\exists \delta > 0$ such that

$$d(x, y) < \delta \Rightarrow \rho(f(x) - f(y)) < \epsilon. \tag{1.33}$$

If f is continuous for all $x \in M$, we say f is continuous on M.

Definition 1.32 Pre-Image

For $A \subseteq N$, the **pre-image** of f is

$$f^{-1}(A) = \{x \in M \mid f(x) \in A\}. \tag{1.34}$$

Theorem 1.8

Given $f : (M, d) \to (N, \rho)$, the following statements are equivalent:

1. *f is continuous on M.*
2. *$\forall x \in M$, if $x_n \to x$ in (M, d) then $f(x_x) \to f(x)$ in (N, ρ)*
3. *If E is closed in N, $f^{-1}(E)$ is closed in M.*
4. *if V is open in N, $f^{-1}(V)$ is open in M.*

Theorem 1.9

Let $f : L \to M$ and $g : M \to N$. If f is continuous at $x \in L$ and g is continuous $f(x) \in M$, $f \circ g : L \to N$ is continuous at $x \in L$.

Definition 1.33 Lipschitz

A function $f : (M, d) \to (N, \rho)$ is **Lipschitz continuous** if $\exists K < \infty$ such that $\rho(f(x), f(y)) \leq Kd(x, y)$ for all $x, y \in M$.

1.3.2 Homeomorphisms

Definition 1.34

The metric spaces (M, d) and (N, ρ) are **homeomorphic** if there exists a bijection $f : (M, d) \to (N, \rho)$ such that f and f^{-1} are continuous on M and N, respectively.

Definition 1.35

Two metrics d and ρ on M are **equivalent** if

$$d(x_n, x) \to 0 \iff \rho(f(x_n), f(x)) \to 0 \text{ as } n \to 0. \tag{1.35}$$

Corollary 1.3

Two metrics d and ρ are ***equivalent*** *if (M, d), (M, ρ) have convergent sequences which converge to the same limit:*

$$x_n \xrightarrow{d} x \iff x_n \xrightarrow{\rho} x \tag{1.36}$$

Theorem 1.10

Let $f : (M, d) \to (N, \rho)$ be a bijection. The following statements are equivalent:

1. *f is a homeomorphism*
2. $x_n \xrightarrow{d} x \iff f(x_n) \xrightarrow{\rho} f(x)$
3. *G is open in $M \iff f(E)$ is open in N.*
4. *E is closed in $M \iff f(E)$ is closed in N.*
5. *$\hat{d}(x, y) = \rho(f(x), f(y))$ if equivalent to d.*

☞ **Remark**
$(\mathbb{R}, \|\cdot\|_1)$, $(\mathbb{R}, \|\cdot\|_2)$,$(\mathbb{R}, \|\cdot\|_\infty)$ are all homeomorphic.

1.4 Connected Sets

Definition 1.36

A metric space M is **disconnected** if it can be written as the union of two non-empty, disjoint, open sets.

$$\begin{aligned} M &= A \cup B \\ A \neq \varnothing&, B \neq \varnothing \\ A \cap B &= \varnothing \end{aligned} \tag{1.37}$$

Definition 1.37 Clopen Sets

A set which is both closed and open is said to be **clopen**.

☞ **Remark**

$$M \text{ is disconnected } \iff \exists A \subset M \text{ such that A is clopen} \tag{1.38}$$

☞ **Remark**
Let $E \subset M$.

$$E \text{ is a disconnected subset of } M \iff \exists U, V \subset M \text{ such that } E = (E \cap U) \cup (E \cap V) \tag{1.39}$$

Where U, V are open in M and satisfy:

1. $(E \cap U) \neq \varnothing$
2. $(E \cap V) \neq \varnothing$

3. $(E \cap U) \cap (E \cap V) = \varnothing$

Theorem 1.11 Intermediate Value Theorem

A subset $E \subseteq \mathbb{R}$ containing more than 1 point is connected if and only if, $\forall x, y \in E$ satisfying $x < y$, we have $[x, y] \subseteq E$.

Corollary 1.4

A subset $E \subseteq \mathbb{R}$ is connected if and only if it is an interval.

Theorem 1.12

A metric space M is disconnected if and only if there exists a continuous map from M on $(\{0, 1\}, d)$, where d is the discrete metric.

1.5 Completeness

1.5.1 Totally Bounded Sets

Theorem 1.13

*A set A in (M, d) is **totally bounded** if and only if, given any $\epsilon > 0$, there exists finitely many points $x_1, x_2, \dots, x_n \in M$ such that*

$$A \subseteq \bigcup_{i=1}^{n} B_\epsilon(x_i) \tag{1.40}$$

Corollary 1.5

*A set A in (M, d) is **totally bounded** if and only if, given any $\epsilon > 0$, there exists finitely many set $A_1, A_2, \dots, A_n \subseteq A$ with $\text{diam}(A_i) < \epsilon$ for $i = 1, 2, \dots, n$ such that*

$$A \subseteq \bigcup_{i=1}^{n} A_i \tag{1.41}$$

☞ **Remark**

Totally bounded $\Rightarrow$ bounded, but Bounded $\not\Rightarrow$ totally bounded.

1.5.2 Totally Bounded Sets vs. Cauchy Sequences

Theorem 1.14

Let (x_n) be a sequence in a metric space and let

$$A = \{x_n \mid n \geq 1\} \tag{1.42}$$

1. *if (x_n) is a Cauchy Sequence, A is totally bounded.*
2. *If A is totally bounded, (x_n) has a Cauchy subsequence.*

1.5.3 Complete Metric Spaces

Definition 1.38

(M, d) is **complete** if every Cauchy sequence in M converges to a point in M.

Theorem 1.15

Let (M, d) be a complete metric space and let A be a subset of M. (A, d) is complete if and only if A is closed in M.

Theorem 1.16 Nested Set Theorem

For a metric space (M, d), the following statements are equivalent:

1. *(M, d) is complete*
2. *let (F_n) be a sequence of closed, non-empty sets satisfying*

$$F_1 \supseteq F_2 \supseteq F_3 \supseteq \dots \tag{1.43}$$

such that $diam(F_n) \to 0$ as $n \to \infty$. Then

$$\bigcap_{n=1}^{\infty} F_n \neq \emptyset. \tag{1.44}$$

3. *Every infinite, totally bounded subset of M has a limit point in M.*

Theorem 1.17

ℓ_2 is complete.

1.5.4 Banach Spaces

Definition 1.39

A complete, normed, linear space is a **Banach Space**.

Definition 1.40 Strict Contraction

Let (M, d) be a metric space and define $f : M \to M$. This f is a **strict contraction** if $\exists \alpha < 1$ such that $d(f(x), f(y)) \leq \alpha d(x, y)$ for all $x, y \in M$.

☞ **Remark**
A contraction $f : M \to M$ is automatically continuous.

Theorem 1.18 Banach Fixed Point

Let (M, d) be complete and $f : M \to M$ be a strict contraction. The $\exists! x \in M$ such that $f(x) = x$. x is called a ***fixed point*** *in M.*
Moreover, given any $x \in M$, the sequence $\left(f^n(x_\circ)\right)_{n=1}^{\infty}$ converges to the fixed point $x = f(x)$ as $n \to \infty$.

1.5.5 Completions

Definition 1.41 Isometry

$f : (M, d) \to (N, \rho)$ is an **isometry** if it satisfies:

$$\rho(f(x), f(y)) = d(x, y). \tag{1.45}$$

In other words, **isometries** preserve distances.

Definition 1.42

A metric space $(\hat{M}, \hat{d})$ is a completion of (M, d) if:

1. $(\hat{M}, \hat{d})$ is complete.
2. (M, d) is isometric to a dense subset of $(\hat{M}, \hat{d})$.

☞ **Remark**
If M is dense in $\hat{M}$, $(\hat{M}, \hat{d})$ is a completion of (M, d).

Theorem 1.19

Every metric space (M, d) has a completion. Moreover, if $(\hat{M}_1, \hat{d}_1)$ and $(\hat{M}_2, \hat{d}_2)$ are both completions of (M, d), then $f : (\hat{M}_1, \hat{d}_1) \to (\hat{M}_2, \hat{d}_2)$ is an isometry.

1.6 Compactness

Definition 1.43

A metric space (M, d) is **compact** if it is both totally bounded and complete.

☞ **Remark** Heine-Borel
A subset $K \subseteq \mathbb{R}$ is compact if and only if K is closed.
Additionally, K is totally bounded if and only iff K is bounded.
So K is compact if and only if it is closed and bounded.

Theorem 1.20

(M, d) is compact if and only if every sequence in M has a subsequence that converges to a point in M.

Corollary 1.6

compact $\Rightarrow$ closed

Corollary 1.7

compact $\Rightarrow$ bounded

Corollary 1.8

Closed subsets of compact metric spaces are compact.

Theorem 1.21

Let $f : (M, d) \to (N, \rho)$ be continuous on M. If K is compact in M, then $f(K)$ is compact in N.

Theorem 1.22 Extreme Value Theorem

Let (M, d) be a complete metric space. and let $f : M \to \mathbb{R}$ be continuous. Then $f(M)$ is bounded and achieves its maximum and minimum values.

Corollary 1.9

If $f : [a, b] \to M$ is continuous, then $\exists c, d, \in \mathbb{R}$ with $c < d$ such that $f([a, b]) = [c, d]$.

Theorem 1.23

In a metric space (M, d), the following are equivalent:

1. *If $\mathcal{G}$ is any collection of open sets in M and $M \subseteq \bigcup\{G : G \in \mathcal{G}\}$, then there exists $G_1, \dots, G_n$ such that*
$$M \subseteq \bigcup_{i=1}^{n} G_i$$
In other words, every open cover of M has a finite subcover.
2. *If $\mathcal{F}$ is any collection of closed sets in M with $\bigcap_{i=1}^{n} F_i \neq \varnothing$, then*
$$\bigcap\{F : F \in \mathcal{F}\} \neq \varnothing.$$

1.7 Uniform Continuity

Definition 1.44

$f : (M, d) \to (N, \rho)$ is **uniformly continuous** if, given any $\epsilon > 0$, there exists $\delta > 0$ such that, $\forall x, y \in M$ with $d(x, y) < \delta$,
$$\rho(f(x), f(y)) < \epsilon$$

☞ **Remark**

Lipschitz functions are uniformly continuous. Given any $\epsilon > 0$, choose $\delta < \dfrac{\epsilon}{K}$ where K is the Lipschitz constant.

Theorem 1.24

If $f : M \to N$ is uniformly continuous and M is totally bounded, then N is also totally bounded. (Uniformly continuous functions map totally bounded sets to totally bounded sets).

Theorem 1.25

If M is compact and $f : M \to N$ is continuous, then f is uniformly continuous.

Theorem 1.26

Assume $(V, \|\cdot\|)$ and $(W, |||\cdot|||)$ are normed linear spaces and consider the map $T : V \to W$, where T is linear, i.e. T satisfies:

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$$

for all $x, y \in M$ and for all scalars α, β.
Then the following are equivalent:

1. T *is Lipschitz:*
   $\exists c > 0$ such that, $\forall x, y \in V$
   $$|||T(x) - T(y)||| \le c\|x - y\|$$
2. T *is uniformly continuous*
3. T *is continuous on* V.
4. T *is continuous at* $\mathbf{0} \in V$.
5. $\exists c > 0$ *such that*
   $$|||T(x)||| \le c\|x\|$$

Definition 1.45

A linear map $T : (V, \|\cdot\|) \to (W, |||\cdot|||)$ is **bounded** if $\exists c > 0$ such that

$$|||T(x)||| \le c\|x\|$$

Definition 1.46

We denote the set of all bounded, linear mappings from V to W as $\mathbf{B(V, W)}$.

Theorem 1.27

$B(V, W)$ is a normed linear space.

Definition 1.47

Let $T \in B(V, W)$. We define the norm of T (known as the **Operator Norm**) as follows:

$$\begin{aligned}\|T\|_{B(V,W)} &= \inf\{c \ge 0 : |||T(x)||| \le c\|x\|, \forall x \in V\} \\ &= \sup_{x \in V, \|x\| \ne 0} \frac{|||T(x)|||}{\|x\|} \\ &= \sup_{\|x\| \le 1} |||T(x)|||\end{aligned}$$

☞ Remark
For all $x \in V$:

$$|||T(x)||| \leq \|T(x)\|_{B(V,W}\|x\|$$

1.8 Sequences of Functions

1.8.1 Pointwise vs. Uniform Convergence

Definition 1.48

Let X be a set and (Y, ρ) a metric space. Let $f : X \to Y$ and $(f_n)_{i=1}^{\infty}$ be a sequence of functions such that $f_n : X \to Y$ for all $n \in \mathbb{N}$.
We say (f_n) converges to f **point-wise** on X if, for every $\hat{x} \in X$,

$$f_n(\hat{x}) \xrightarrow{\rho} f(\hat{x})$$

Definition 1.49

We say (f_n) is **uniformly convergent** if, given any $\epsilon > 0$ and $x \in X$, there exists $N \in \mathbb{N}$ such that, for all $n \geq N$,

$$\rho(f_n(x), f(x)) < \epsilon$$

for each $\epsilon > 0$.

Theorem 1.28

Let (X, d) and (Y, ρ) be metric spaces and $f_n : X \to Y \ \forall n \in \mathbb{N}$. Asssume $f_n \to f$ uniformly on X and f_n is continuous at $x \in X \ \forall n \in \mathbb{N}$. Then f is also continuous at x.

Theorem 1.29

Suppose $f_n : [a, b] \to \mathbb{R}$ is continuous $\forall n \in \mathbb{N}$ and assume $f_n \to f$ uniformly on $[a, b]$. Then

$$\int_a^b f_n \to \int_a^b f$$

1.8.2 Space of Bounded Functions

Definition 1.50

Given a set X, let $B(X)$ denote the space of all real valued, bounded functions on X. So $f \in B(X)$ means $f : X \to \mathbb{R}$ and $\sup_{x \in X} |f(x)| < \infty$. We equip $B(X)$ with the sup norm: $\|f\|_{B(X)} = \|f\|_{\infty} = \sup_{x \in X} |f(x)|$

☞ Remark
$\|\cdot\|_{\ell_\infty}$ refers specifically to sequences.

Remark
If $f_n \to f$ in $B(X)$, or $\|f_n - f\|_\infty \to 0$ as $n \to \infty$, then, given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $\|f_n - f\|_\infty = \sup_{x \in X} |f_n(x) - f(x)| < \epsilon$. But then $\forall n \geq N$ and $\forall x \in X$, $|f_n(x) - f(x)| < \epsilon$, so $f_n \to f$ uniformly on X.

Theorem 1.30

$B(X)$ is complete under the sup norm. This means, given any Cauchy sequence $(f_n) \in B(X)$, $f_n \to f \in B(X)$. Moreover, $\exists c > 0$ such that $\|f_n\|_\infty \leq C$ for all $n \in \mathbb{N}$ and $\|f_n\|_\infty \to \|f\|_\infty$

Definition 1.51

A Cauchy sequence $(f_n) \in B(X)$ is called **Uniformly Cauchy**.

Definition 1.52

A bounded sequence in $B(X)$ is called **Uniformly Bounded**.

Theorem 1.31

Assume X is a coompact metric space. Then $C_b(x) = C(x)$. If X is compact and $f : X \to \mathbb{R}$ is continuous, then $f(x)$ is compact in $\mathbb{R}$ so $f(x)$ is bounded. Therefore, $C(x) = c_b(x)$.

1.9 Equicontinuity

Remark
If $f \in C(X)$ and X is compact, then f is uniformly continuous.

Definition 1.53

Let $\mathcal{F}$ be a collection of real valued function on a metric space X. We say $\mathcal{F}$ is **equicontinuous** if, given any $\epsilon > 0$, there exists $\delta > 0$ such that, $\forall x, y \in X$ with $d(x, y) < \delta$, $|f(x) - f(y)| < \epsilon$ for all $f \in \mathcal{F}$.

Theorem 1.32

Let X be a compact set. Any finite subset of $C(X)$ is equicontinuous.

Definition 1.54

Fix $k > 0$ and $\alpha > 0$. Consider the set of $\{f \in C([0,1]) : |f(x) - f(y)| \leq k|x - y|^\alpha, \forall x, y \in [0,1]\}$. We call this set **Lip**$_k^\alpha$.

Theorem 1.33

Given $\epsilon > 0$, choose $\delta = (\frac{\epsilon}{k})^\alpha$. Then Lip_k^α is equicontinuous

Definition 1.55

A collection of real valued functions $\mathcal{F}$ on X is **uniformly equibounded** if $\{f(x) : x \in X, f \in \mathcal{F}\}$ is a bounded set in $\mathbb{R}$

$$\sup_{x\in X, f\in\mathcal{F}} |f(x)| = \sup_{f\in\mathcal{F}} \|f\|_\infty < \infty$$

1.10 Arzela-Ascoli Theorem

Definition 1.56 Uniformly Bounded

A collection of real values functions $\mathcal{F}$ on a set X is is **Uniformly Bounded** if

$$\{f(x) : x \in X, f \in \mathcal{F}\}$$

or

$$\sup_{f\in\mathcal{F}, x\in X} |f(x)| = \sup_{f\in\mathcal{F}} \|f\|_\infty < \infty$$

or

$$\exists C > 0 \text{ such that } \|f\|_\infty \le C$$
$$\forall f \in \mathcal{F}$$

☞ **Remark**

For $\mathcal{F} \subseteq C(X)$, where $C(X)$ is equipped with $\|\cdot\|_\infty$, $\mathcal{F}$ is uniformly bounded if and only if $\mathcal{F}$ is a bounded subset of $C(X)$.

Theorem 1.34 Arzela-Ascoli

Let X be a compact metric space and let $\mathcal{F} \subseteq C(X)$. $\mathcal{F}$ is compact if and only if $\mathcal{F}$ is closed, uniformly bounded, and equicontinuous.

Corollary 1.10

Let X be a compact metric space. If (f_n) is uniformly bounded and equicontinuous on $C(X)$, then there exists a subsequence of (f_n) that converges uniformly on X.

Chapter 2 Measure Theory (MTH 512)

2.1 Riemann Integral

Definition 2.1

Let P be a **partition** of $[a, b]$,

$$P = x_0, x_1, \ldots, x_n$$

such that

$$a = x_0 < x_1 < \ldots < x_n = b$$

Definition 2.2

Assume $f : [a, b] \to \mathbb{R}$ is bounded.

$$L(f, P, [a, b]) = \sum_{j=1}^{n} (\inf_{x \in [x_{j-1}, x_j]} f(x))(x_j - x_{j-1})$$

is the **Lower Riemann Sum** of f.

Definition 2.3

Assume $f : [a, b] \to \mathbb{R}$ is bounded.

$$U(f, P, [a, b]) = \sum_{j=1}^{n} (\sup_{x \in [x_{j-1}, x_j]} f(x))(x_j - x_{j-1})$$

is the **Upper Riemann Sum** of f.

Definition 2.4

$$L(f, [a, b]) = \sup_{P} L(f, P, [a, b])$$

is the **Lower Riemann Integral** of f.

Definition 2.5

$$U(f, [a, b]) = \inf_{P} U(f, P, [a, b])$$

is the **Upper Riemann Integral** of f.

Definition 2.6

A bounded function $f : [a, b] \to \mathbb{R}$ is **Riemann Integrable** on $[a, b]$ if

$$L(f, [a, b]) = U(f, [a, b])$$

Theorem 2.1

If $f : [a, b] \to \mathbb{R}$ is continuous, f is Riemann Integrable.

2.2 Measures

2.2.1 Outer Measures

Definition 2.7

If I is an open interval in $\mathbb{R}$ with $a < b$ (i.e. $I = (a,b), I = (-\infty, a), I = (a, \infty)$, or $I = (-\infty, \infty)$). The **length** of I is given by

$$\ell(I) = \begin{cases} b-a, I = (a,b) \\ \infty, I = (-\infty, a), I = (a, \infty), I = (-\infty, \infty) \\ 0, I = \varnothing \end{cases}$$

Definition 2.8

For $A \subseteq \mathbb{R}$, the **Outer Measure** of A is

$$|A| = \inf\{\sum_{k=1}^{\infty} \ell(I_k) : A \subseteq \bigcup_{k=1}^{\infty} I_k\}$$

Where $\{I_k\}_{k=1}^{\infty}$ is a collection of open intervals and $|A|$ is the infimum over all such collections.

Theorem 2.2

The outer measure of any countable subset of $\mathbb{R}$ is 0.

Theorem 2.3

Suppose $A \subseteq B \subseteq \mathbb{R}$, then $|A| \leq |B|$.

Theorem 2.4

Assume $t \in \mathbb{R}$ and $A \subseteq \mathbb{R}$, then $|t + A| = |A|$, where

$$t + A = \{t + a : a \in A\}$$

Theorem 2.5

Suppose $\{A_1, A_2, A_3, \dots\}$ is a countable collection of subsets of $\mathbb{R}$. Then

$$|\bigcup_{k=1}^{\infty} A_k| \leq \sum_{k=1}^{\infty} |A_k|$$

☞ **Remark**

$\exists A_1, A_2 \in \mathbb{R}$ with $A_1 \bigcap A_2 = \varnothing$ such that

$$|A_1 \cup A_2| \neq |A_1| + |A_2|$$

Theorem 2.6

Let $a, b \in \mathbb{R}$, $a < b$. Then

$$|[a,b]| = b - a$$

Theorem 2.7

$\nexists$ a function μ with all the following properties:

1. *μ maps all subsets of $\mathbb{R}$ to $[0, \infty]$.*
2. *$\mu(I) = \ell(I)$ for all open intervals $I \in \mathbb{R}$.*
3. *$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k)$ for all $\{A_k\}_{k=1}^{\infty}$ (pairwise disjoint).*
4. *$\mu(t + A) = \mu(A)$ for all $t \in \mathbb{R}$, $A \subseteq \mathbb{R}$.*

2.2.2 σ-algebras

Definition 2.9 σ-algebra

Let X be a set and $\mathscr{S}$ be a collection of subsets of X. Then $\mathscr{S}$ is a **σ-algebra** on X if:

1. $\varnothing \in \mathscr{S}$
2. If $E \in \mathscr{S}$ then $X - E \in \mathscr{S}$
3. If $\{E_k\}_{k=1}^{\infty}$ is a collection in $\mathscr{S}$ then

$$\bigcup_{k=1}^{\infty} E_k \in \mathscr{S}$$

☞ **Remark**

Suppose $\mathscr{S}$ is a σ-algebra on X, then

1. $X \in \mathscr{S}$
2. $D, E \in \mathscr{S} \Rightarrow D \cap E \in \mathscr{S}$ and $D \cup E \in \mathscr{S}$ and $D - E \in \mathscr{S}$
3. If $\{E_k\}_{k=1}^{\infty}$ is a countable collection in $\mathscr{S}$, then $\bigcap_{k=1}^{\infty} E_k \in \mathscr{S}$

2.2.3 Measurable Spaces

Definition 2.10

A **measurable space** is an ordered pair $(X, \mathscr{S})$, where X is a set and $\mathscr{S}$ is a σ-algebra on X. An element of $\mathscr{S}$ is said to be $\mathscr{S}$ **measurable**.

☞ **Remark**

Consider $X = \mathbb{R}$. Let $\mathscr{S}$ be the collection of all sets E such that E or $X - E$ is countable.

1. $\mathbb{Q}$ is $\mathscr{S}$ measurable
2. $\mathbb{R} - \mathbb{Q}$ is $\mathscr{S}$ measurable
3. $(0, 1)$ is <u>not</u> $\mathscr{S}$ measurable

2.2.4 Borel Subsets

Theorem 2.8

Let X be a set and let $\mathscr{A}$ be a collection of subsets of X. Then the intersection of all σ-algebras on X which contain $\mathscr{A}$ is also a σ-algebra containing $\mathscr{A}$. Furthermore, the intersection is the smallest possible σ-algebra containing $\mathscr{A}$.

Definition 2.11

The smallest σ-algebra on $\mathbb{R}$ containing all open subsets of $\mathbb{R}$ is called the collection of **Borel Subsets**. An element of this σ-algebra is called a **Borel Set**.

☞ **Remark**

1. Open sets are Borel Sets
2. Closed sets are Borel Sets
3. $[a, b)$,$(a, b]$ are Borel Sets
4. x is a Borel Set
5. Countable subsets of $\mathbb{R}$ are Borel Sets
6. $\mathbb{Q}$ and $\mathbb{R} - \mathbb{Q}$ are Borel Sets
7. Any countable union of countable intersection of (1)-(7) is a Borel Set

2.2.5 Measures

Definition 2.12

let X be a set and $\mathscr{S}$ be a σ-algebra on X, then $(X, \mathscr{S})$ is a measurable space. A **measure** on $(X, \mathscr{S})$ is a function $\mu : \mathscr{S} \to [0, \infty]$ such that:

1.
$$\mu(\emptyset) = 0$$

2.
$$\mu\Big(\bigcup_{k=1}^{\infty} E_k\Big) = \sum_{k=1}^{\infty} \mu(E_k)$$

☞ **Remark**

Let $X = \mathbb{R}$ and $\mathscr{S} = P(X)$, then $(X, \mathscr{S})$ is a measurable space but $\mu = |\cdot|$ is not a measure on $(X, \mathscr{S})$ because (2) fails.

Definition 2.13 Counting Measure

Let X be a set and $\mathscr{S} = P(X)$. Define $\mu : \mathscr{S} \to [0, \infty]$ as

$$\mu(E) = \begin{cases} +\infty, & E \in \mathscr{S} \text{ is infinite.} \\ n, & E \in \mathscr{S} \text{ is finite.} \end{cases}$$

where n is the number of elements in $\mathscr{S}$.

☞ Remark

Consider the set $X = \{1, 2, 3, 4, \ldots, N-1, N\}$ and $\mathscr{S} = P(X)$ and let μ be a counting measure on $(X, \mathscr{S})$. Consider a sum of real numbers $a_1 + a_2 + a_3 + a_4 + \ldots + a_N$. Let $f(k) = a_k$ for each $1 \leq k \leq N$ ($f : X \to \mathbb{R}$). Then

$$\begin{aligned} \sum_{k=1}^{N} a_k &= \sum_{k=1}^{N} f(k) \\ &= \sum_{k=1}^{N} f(k) \cdot \mu(\{k\}) \\ &= \int_X f \cdot d\mu \end{aligned}$$

Definition 2.14

A **Measure Space** $(X, \mathscr{S}, \mu)$ is a measurable space with a measure on it.

Theorem 2.9

Suppose $(X, \mathscr{S}, \mu)$ is a measure space. Let $D, E \in \mathscr{S}$ such that $D \subseteq E$, then

1. $\mu(D) \leq \mu(E)$
2. $\mu(D - E) = \mu(D) - \mu(E)$

Theorem 2.10 Countable Subadditivity

Let $(X, \mathscr{S}, \mu)$ be a measure space and $E_1, E_2, E_3, \ldots \in \mathscr{S}$ (not necessarily disjoint), then

$$\mu\Big(\bigcup_{k=1}^{\infty} E_k\Big) \leq \sum_{k=1}^{\infty} \mu(E_k)$$

Theorem 2.11

Let $(X, \mathscr{S}, \mu)$ be a measurable space. Let $E_1 \subseteq E_2 \subseteq E_3 \subseteq \ldots$ be a nested sequence of sets in $\mathscr{S}$, then

$$\mu\Big(\bigcup_{k=1}^{\infty} E_k\Big) = \lim_{k\to\infty} \mu(E_k)$$

Theorem 2.12

Let $(X, \mathscr{S}, \mu)$ be a measurable space. Let $(X, \mathscr{S}, \mu)$ be a measurable space. Let $E_1 \supseteq E_2 \supseteq E_3 \supseteq \ldots$ be a nested sequence of sets in $\mathscr{S}$ and $\mu(E_1) < \infty$, then

$$\mu\Big(\bigcap_{k=1}^{\infty} E_k\Big) = \lim_{k\to\infty} \mu(E_k)$$

Theorem 2.13

Assume $(X, \mathscr{S}, \mu)$ is a measure space and $D, E \in \mathscr{S}$ with $\mu(D \cup E) < \infty$. Then $\mu(D\cup)E = \mu(D) + \mu(E) - \mu(D \cap E)$.

2.3 Lebesgue Measure

☞ **Remark**
In constructing the Lebesgue Measure, the idea is to show that the outer measure, when restricted to $(\mathbb{R}, \mathbb{B})$ where $\mathbb{B}$ is the Borel Set of $\mathbb{R}$, is a measure. In other words, $(\mathbb{R}, \mathbb{B}, |\cdot|)$ is a measure space.

Theorem 2.14
Let $A, G \subseteq \mathbb{R}$, $A \cap G = \varnothing$ and G open. Then $|A \cup G| = |A| + |G|$.

Theorem 2.15
Let $A, F \subseteq \mathbb{R}$, $A \cap G = \varnothing$ and F open. Then $|A \cup F| = |A| + |F|$.

Theorem 2.16
Let $B \subseteq \mathbb{R}$ be a Borel set. The $\forall \epsilon > 0$, there exists a closed set $F \subseteq B$ such that $|B - F| < \epsilon$.

Theorem 2.17
Suppose $A, B \subseteq \mathbb{R}$, $A \cap B = \varnothing$, and B is a Borel Set. Then

$$|A \cup B| = |A| + |B|$$

Theorem 2.18
Outer Measure is a measure on the measurable space $(\mathbb{R}, \mathscr{B})$ where $\mathscr{B}$ is the set of all Borel Sets. So $(\mathbb{R}, \mathscr{B}, |\cdot|)$ is a measure space.

Definition 2.15 Lebesgue Measure
Lebesque Measure is the measure on $(\mathbb{R}, \mathcal{B})$ which assigns to each Borel set its outer measure.

2.3.1 Lebesgue Measurable Sets

Definition 2.16
If $A \subseteq \mathbb{R}$, A is **Lebesgue Measurable** if $\exists$ a Borel set $B \subseteq A$ such that $|A - B| = \varnothing$.

Definition 2.17
Let $A \subseteq \mathbb{R}$. The following statements are equivalent:

1. A is Lebesgue Measurable.
2. $\forall \epsilon > 0, \exists F$ closed in A such that $|A - F| < \epsilon$.
3. $\exists$ sequence of closed sets $F_1, F_2, F_3, \ldots \subseteq A$ such that

$$\left| A - \bigcup_{i=1}^{\infty} F_i \right| = 0$$

4. $\forall \epsilon > 0, \exists G$ open with $G \supseteq A$ such that $|G - A| < \epsilon$.
5. $\exists$ sequence of open sets $G_1, G_2, G_3, \ldots \supseteq A$ such that

$$\left| \left(\bigcap_{i=1}^{\infty} G_i \right) - A \right| = 0$$

6. $\exists$ a Borel set $B \supseteq A$ such that $|B - A| = 0$

Theorem 2.19

Outer Measure is a measure on $(\mathbb{R}, \mathcal{L})$, where $\mathcal{L}$ is the σ-algebra of Lebesgue measurable sets.

Definition 2.18 Alternative Definition of Lebesgue Measure

Lebesgue Measure is the measure on $(\mathbb{R}, \mathcal{L})$ which assigns to each $A \in \mathcal{L}$ its outer measure.

☞ **Remark**

The two definitions of Lebesgue Measure are not equivalent, however

$$\forall A \in \mathcal{L},$$
$$A = B \cup (A - B)$$

where B is Borel and $|A - B| = 0$. So, in practice, the difference in definition doesn't matter.

Theorem 2.20

Every set A with $|A| = 0$ is Lebesgue measurable.

☞ **Remark**

For any Lebesgue measurable set A,

$$A = B \cup (A - B)$$

where B is Borel and $|A - B| = 0$. So $\mathcal{L}$ is the smallest σ-algebra containing the Borel sets and the sets of outer measure 0. (Note: non-Borel sets of outer measure 0 do exists, but they don't really matter for any reason.)

2.4 Measurable Functions

Definition 2.19

Suppose $(X, \mathscr{S})$ is a measurable space. A function $f : X \to \mathbb{R}$ is a **measurable function** if $f^{-1}(B) \in \mathscr{S}$ for all $B \in \mathbb{B}$.

2.4.1 Characteristic Functions

Definition 2.20

Let X be a set and $E \subseteq X$. The **characteristic function** of E, $\chi_E : X \to \mathbb{R}$, is defined by:

$$\chi_E(x) = \begin{cases} 1, & x \in E \\ 0, & x \notin E \end{cases}$$

Theorem 2.21

Suppose $(X, \mathscr{S})$ is a measurable space. If $E \subseteq X$, then chi_E is measurable iff $E \subseteq \mathscr{S}$ (i.e. E is $\mathscr{S}$ measurable).

Definition 2.21

Suppose $X \subseteq \mathbb{R}$, then $f : X \to \mathbb{R}$ is **Borel Measurable** if $f^{-1}(B)$ is a Borel set $\forall B \in \mathbb{B}$.

Definition 2.22

Suppose $A \subseteq \mathbb{R}$. Then $f : A \to \mathbb{R}$ is **Lebesgue Measurable** if $f^{-1}(B)$ is Lebesgue Measurable for all Borel sets.

Theorem 2.22

Suppose $(X, \mathscr{S})$ is a measurable space and $f : X \to \mathbb{R}$, then f is measurable iff $f^{-1}(A) \in \mathscr{S}$ for all open sets $A \subseteq \mathbb{R}$.

Theorem 2.23

Suppose $(X, \mathscr{S})$ is a measurable space and $f : X \to \mathbb{R}$, then f is measurable iff $f^{-1}((a, \infty)) \in \mathscr{S}$ for all $a \in \mathbb{R}$.

Theorem 2.24

Suppose $(X, \mathscr{S})$ is a measurable space and let $f_1, f_2, f_3, \ldots$ be a sequence of measurable functions with $f_k : X \to \mathbb{R}$ for all k. Suppose, for all $x \in X$, $\lim_{k\to\infty} f_k(x)$ exists. Let

$$f = \lim_{k\to\infty} f_k(x)$$

for all $x \in X$. Then f is also measurable.

Corollary 2.1

Suppose $(X, \mathscr{S})$ is a measurable space and let $f_1, f_2, f_3, \ldots$ be a sequence of measurable functions with $f_k : X \to \mathbb{R}$ for all k. Suppose, for all $x \in X$, $\lim_{k\to\infty} f_k(x)$ exists. Then for any $a \in \mathbb{R}$

$$f^{-1}((a, \infty)) = \bigcup_{j=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{m=k}^{\infty} f_k^{-1}((a + \frac{1}{j}, \infty)) \in \mathscr{S}$$

Theorem 2.25

If $f : X \to \mathbb{R}$ is continuous with $X \subseteq \mathbb{R}$, then f is both Borel and Lebesgue measurable.

2.4.2 Composition of Measurable Functions

Theorem 2.26

Let $(X, \mathscr{S})$ be a measurable space and $f : X \to \mathbb{R}$ be $\mathscr{S}$ measurable. Assume $Y \subseteq f(X)$ and let $g : Y \to \mathbb{R}$ be Borel measurable. then $g \circ f : X \to \mathbb{R}$ is $\mathscr{S}$ measurable.

Example 2.1

Assume f is $\mathscr{S}$ measurable. Then $f^2, \frac{1}{2}f, -f, |f|$ are $\mathscr{S}$ measurable.

Theorem 2.27

Suppose $(X, \mathscr{S})$ is a measurable set. Let $f, g : X \to \mathbb{R}$ be $\mathscr{S}$ measurable. Then the following are also $\mathscr{S}$ measurable:

1. $f + g$
2. $f - g$
3. fg
4. f/g *(If $g(x) \neq 0, \forall x \in X$)*

2.4.3 Convergence of Measurable Functions

Theorem 2.28 Egorov's

Suppose $(X, \mathscr{S}, \mu)$ is a measure space with $\mu(x) < \infty$. Let $\{f_k\}$ be a sequence of measurable functions $f_k : X \to \mathbb{R}$ for all k, with $f_k \to f$ for all $x \in X$ (pointwise). Then $\forall \epsilon > 0, \exists E \in \mathscr{S}$ such that $\mu(X - E) < \epsilon$ and $f_k \to f$ uniformly on E.

Remark
We can assume $f_k \to f$ pointwise "almost everywhere", meaning everywhere except on a subset $A \subseteq X$ with $\mu(A) = 0$.

2.4.4 Simple Functions

Definition 2.23

A subset $A \subseteq [-\infty, \infty]$ is called a **Borel Set** if $A \cap \mathbb{R}$ is a Borel set of $\mathbb{R}$.

Remark
The set of Borel Sets of $[-\infty, \infty]$ is a σ-algebra on $[-\infty, \infty]$.

Definition 2.24

Let $(X, \mathscr{S})$ be a measurable space. Then $f : X \to [-\infty, \infty]$ is $\mathscr{S}$ measurable if $f^{-1}(B) \in \mathscr{S}$ for all Borel Sets B in $[-\infty, \infty]$.

Theorem 2.29

Suppose $(X\mathscr{S})$ is a measurable space. Then $f : X \to [-\infty, \infty]$ is $\mathscr{S}$ measurable if and only if $f^{-1}((a, \infty]) \in \mathscr{S}$ for all $a \in \mathbb{R}$.

Definition 2.25

A function if called **simple** if it takes on finitely many values in $\mathbb{R}$
Let $(X, \mathscr{S})$ be a measurable space. Let $f : X \to \mathbb{R}$ be a simple function on the non-zero values

$c_1, c_2, c_3, \ldots, c_n$. Then

$$f = c_1\chi_{E_1} + c_2\chi_{E_2} + c_3\chi_{E_3} + \ldots + c_n\chi_{E_n}$$

Where $E_k = f^{-1}(\{c_k\})$ for all $1 \leq k \leq n$.
Note that if f is $\mathscr{S}$ measurable, then $E_k = f^{-1}(\{c_k\}) \in \mathscr{S}$ for all k. If $E_k \in \mathscr{S}$ for all k then ξ_{E_k} is $\mathscr{S}$ measurable, so

$$f = \sum_{k=1}^{n} c_k\chi_{E_k}$$

is $\mathscr{S}$ measurable. fo f is $\mathscr{S}$ measurable if and only if $E_k \in \mathscr{S}$ for all $1 \leq k \leq n$

2.4.5 Approximation by Simple Functions

Theorem 2.30

Let $(X, \mathscr{S})$ be a measurable space and $f : X \to [-\infty, \infty]$ be $\mathscr{S}$-measurable. Then $\exists$ a sequence $f_1, f_2, \ldots, f_k : X \to \mathbb{R}$ for all k such that

1. *Each f_k is a simple function*
2. *$|f_k(x)| \leq |f_{k+1}(x)| \leq |f(x)|$ for all $x \in X$ and $k \in \mathbb{N}$*
3. $\lim\limits_{k\to\infty} f_k(x) = f(x)$
4. *If f is bounded, the $f_k \to f$ uniformly on X.*

Theorem 2.31 Lusin's Theorem

Suppose $g : \mathbb{R} \to \mathbb{R}$ is Borel measurable. Then given $\epsilon > 0$, $\exists$ closed $F \subset \mathbb{R}$ such that $|\mathbb{R} - F| \leq \epsilon$ and $g|_F$ is continuous.

Theorem 2.32

If $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue Measurable, there exists a Borel Measurable $g : \mathbb{R} \to \mathbb{R}$ such that

$$|\{x : g(x) \neq f(x)\}| = 0$$

Theorem 2.33

Let $(X\mathscr{S})$ be a measurable space, $f_1, f_2, \ldots$ be a sequence of $\mathscr{S}$-measurable functions with $f_k : \mathbb{R} \to \mathbb{R}$ for all $k \in \mathbb{N}$, then $\{x \in X : \lim\limits_{k\to\infty} f_k(x)$ exists in $\mathbb{R}\}$

Theorem 2.34

If $f, g : X \to [-\infty, \infty]$ satisfy

$$\mu(\{x \in X : f(x) \neq g(x)\}) = 0$$

where μ is the Lebesgue measure, then we say f and g are equal almost everywhere.

2.5 Lebesgue Integration

Remark
By convention, let

$$\infty \times 0 = 0 \times \infty = 0$$

Definition 2.26

Let $\mathscr{S}$ be a σ-algebra on X, then an $\mathscr{S}$**-partition** on X is a finite collection of disjoint sets $A_1, A_2, \ldots, A_n$ in $\mathscr{S}$ such that

$$\bigcup_{j=1}^{n} A_j = X$$

Definition 2.27

Suppose $(X, \mathscr{S}, \mu)$ is a measure space and let $f : X \to [0, \infty]$ be $\mathscr{S}$-measurable. Let $P = \{A_1, A_2, \ldots, A_n\}$ be an $\mathscr{S}$-partition on X. Then the **Lower Lebesgue Sum** is defined to be

$$\mathcal{L}(f, P) = \sum_{j=1}^{n} \mu(A_j) \inf_{x \in A_j} f(x)$$

Definition 2.28

Suppose $(X, \mathscr{S}, \mu)$ is a measure space and let $f : X \to [0, \infty]$ be $\mathscr{S}$-measurable. The **Integral With Respect To** μ (i.e. **Lebesgue Integration** is defined to be

$$\int_X f d\mu = \sup\{\mathcal{L}(f, P) : P \text{ is a partition on } X\}$$

Remark
Suppose $(X, \mathscr{S}, \mu)$ is a measure space and $E \in \mathscr{S}$. Then

$$\int_X \chi_E d\mu = \mu(E)$$

2.5.1 Integrals of Simple Functions

Theorem 2.35

Suppose $(X, \mathscr{S}, \mu)$ is a measure space and $E_1, E_2, .., E_n$ is a disjoint collection in $\mathscr{S}$. Let $c_1, c_2, \ldots, c_n \in [0, \infty]$. Then

$$\int_X \sum_{k=1}^{n} c_k \chi_{E_k} d\mu = \sum_{k=1}^{n} c_k \mu(E_k)$$

Theorem 2.36 Preservation of Order

Suppose $(X, \mathscr{S}, \mu)$ is a measure space. Let $f, g : X \to [0, \infty]$ be $\mathscr{S}$-measurable. Assume $f(x) \leq$

$g(x)$ for all $x \in X$. Then

$$\int_X f d\mu \leq \int_X g d\mu$$

Theorem 2.37

Suppose $(X, \mathscr{S}, \mu)$ is a measure space and $f : X \to [0, \infty]$ is $\mathscr{S}$-measurable. Then

$$\int_X f d\mu = \sup\Big(\{\sum_{j=1}^{n} c_j \mu(A_j) : \{A_1, A_2, \dots, A_n\} \text{ is a disjoint collection of sets in } \mathscr{S}, \\ c_1, c_2, \dots, c_n \in [0, \infty) \text{ and } f(x) \geq \sum_{j=1}^{m} c_j \chi_{A_j}(x) \forall x \in X\}\Big)$$

2.5.2 Monotone Convergence

Theorem 2.38 Monotone Convergence Theorem

Suppose $(X, \mathscr{S}, \mu)$ is a measure space. Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of functions such that $f_k : X \to [0, \infty]$ is $\mathscr{S}$-measurable for all $k \in \mathbb{N}$ and

$$0 \leq f_1 \leq f_2 \leq \dots$$

for all $x \in X$. Let $f(x) = \lim_{k\to\infty} f_k(x)$. Then

$$\lim_{k\to\infty} \int_X f_k d\mu = \int_X \lim_{k\to\infty} f_k d\mu = \int_X f d\mu$$

Theorem 2.39

Suppose $(X, \mathscr{S}, \mu)$ is a measure space and $E_1, E_2, \dots, E_n \in \mathscr{S}$ are <u>not necessarily disjoint</u> and $c_1, c_2, \dots, c_n \in [0, \infty]$. Then

$$\int_X \sum_{k=1}^{n} c_k \chi_{E_k} d\mu = \sum_{k=1}^{n} c_k \mu(E_k)$$

Theorem 2.40

Suppose $(X, \mathscr{S}, \mu)$ is a measure space. Assume $a_1, a_2, \dots, a_m, b_1, b_2, \dots, b_n \in [0, \infty]$, $A_1, A_2, \dots, A_m, B_1, B_2, \dots, B_n \in \mathscr{S}$ such that

$$\sum_{j=1}^{m} a_j \chi_{A_j} = \sum_{k=1}^{n} b_k \chi_{B_k}$$

Then

$$\sum_{j=1}^{m} a_j \mu(A_j) = \sum_{k=1}^{n} b_k \mu(B_k)$$

Theorem 2.41

Suppose $(X, \mathscr{S}, \mu)$ is a measure space. Let $f, g : X \to [0, \infty]$ be $\mathscr{S}$-measurable. Then

$$\int_X (f+g) d\mu = \int_X f d\mu + \int_X g d\mu$$

Definition 2.29

Let $f : X \to [-\infty, \infty]$. Define:

$$f^+ : X \to [0, \infty]$$

and

$$f^- : X \to [0, \infty]$$

as

$$f^+(x) = \begin{cases} f(x), & f(x) \geq 0 \\ 0, & f(x) < 0 \end{cases}$$

$$f^- = \begin{cases} 0 & f(x) \geq 0 \\ -f(x) & < 0 \end{cases}$$

so

$$f^+ = f\chi_{f^{-1}[0,\infty]}$$

$$f^- = -f\chi_{f^{-1}[-\infty,0]}$$

Remark

If $f : X \to [-\infty, \infty]$ is $\mathscr{S}$-measurable, f^+ and f^- are also $\mathscr{S}$-measurable.

Definition 2.30

Given measurable space $(X, \mathscr{S}, \mu)$ and $\mathscr{S}$-measurable function $: X \to [\infty, \infty]$ such that either

$$\int_X f^+ d\mu < \infty$$

or

$$\int_X f^- d\mu < \infty$$

Then

$$\int_X f d\mu \equiv \int_X f^+ d\mu - \int_X f^- d\mu$$

(Note: otherwise, $\int f d\mu = \infty - \infty$ (undefined))

Remark
Note that

$$\int_X |f|d\mu = \int_X (f^+ + f^-)d\mu = \int_X f^+ d\mu + \int_X f^- d\mu$$

Therefore $\int_X |f|d\mu < \infty \iff \int_X f^+ d\mu < \infty$ and $\int_X f^- d\mu < \infty$

2.5.3 Properties of the Integral

Theorem 2.42

Let $f : X \to [-\infty, \infty]$ be an $\mathscr{S}$-measurable function and $\int_X f d\mu$ be defined. The $\forall c \in \mathbb{R}$,

$$\int_X cf d\mu = c \int_X f d\mu$$

Theorem 2.43

Suppose $f : X \to [-\infty, \infty]$ such that $\int |f|d\mu < \infty$. Then

$$\left| \int f d\mu \right| \leq \int |f|d\mu$$

2.6 Limits of Integrals and Integrals of Limits

Definition 2.31
Let $E \in \mathscr{S}$ and $f : x \to [-\infty, \infty]$ be $\mathscr{S}$-measurable. Define

$$\int_E f d\mu = \int_X \chi_E f d\mu$$

Theorem 2.44 Bounded Convergence Theorem

Assume $\mu(X) < \infty$. Let $f_1, f_2, f_3, \ldots$ be a sequence of $\mathscr{S}$-measurable functions such that $f_k \to f$ pointwise on X and $f_k : X \to \mathbb{R}$ for all $k \in \mathbb{N}$ and $f : X \to \mathbb{R}$. Suppose $\exists c > 0$ such that $|f_k(x)| \leq c$ $\forall x \in X$ and $\forall k \in \mathbb{N}$. Then

$$\lim_{k\to\infty} \int f_k d\mu = \int f d\mu$$

Theorem 2.45

Let $E \in \mathscr{S}$. Assume $f : X \to [-\infty, \infty]$ such that $\int_X |f|d\mu < \infty$. Then

$$\left| \int_E f d\mu \right| \leq \mu(X - E) \sup_{x \in E} |f(x)|$$

Theorem 2.46

Let $e \in \mathscr{S}$ and $g : X \to [0, \infty]$ be $\mathscr{S}$-measurable and assume $\int_X g d\mu < \infty$. Then $\forall \epsilon > 0, \exists \delta > 0$ such that whenever $\mu(E) < \delta$,

$$\int_E g d\mu < \epsilon$$

Definition 2.32

Let $f, g : X \to [-\infty, \infty]$ be $\mathscr{S}$-measurable and assume

$$\mu(\{x \in X : f(x) \neq g(x)\}) = 0$$

Then we say $f = g$ **almost everywhere** on X or $f = g$ **a.e.** on X.

Theorem 2.47

If $f = g$ a.e. on X, then

$$\int_X f d\mu = \int_X g d\mu$$

Theorem 2.48

Let $g : X \to [0, \infty]$ be $\mathscr{S}$-measurable and assume $\int_X |g| < \infty$. Then $\forall \epsilon > 0$, $\exists E \in \mathscr{S}$ with $\mu(E) < \infty$ and

$$\int_{X-E} g d\mu < \epsilon$$

In other words: Integrable functions live mostly on sets of finite measure.

Theorem 2.49 Dominated Convergence Theorem

Let $f : X \to [0, \infty]$ be $\mathscr{S}$-measurable. Let $f_1, f_2, f_3, \ldots$ be a sequence of $\mathscr{S}$-measurable functions such that

$$\lim_{k\to\infty} f_k(x) \to f(x) \text{ a.e. on } X$$

Assume $\exists g : X \to [0, \infty]$ also $\mathscr{S}$-measurable such that:

1) $\int_X g d\mu < \infty$

2) $|f_k(x)| \leq g(x)$ for all $k \in \mathbb{N}$ a.e. on X

Then

$$\lim_{k\to\infty} \int_X f_k d\mu = \int_X f d\mu$$

2.6.1 Approximation by Nice Functions

Definition 2.33

Let $f : X \to [-\infty, \infty]$ be $\mathscr{S}$-measurable. Set

$$\|f\|_1 = \int_X |f| d\mu$$

Then define $\mathscr{L}^1(\mu)$ to be

$$\mathscr{L}^1(\mu) = \{f : X \to [-\infty, \infty] : \int_X |f| d\mu < \infty\}$$

$\mathscr{L}^1$ is referred to as the **Lebesgue Space**.

Theorem 2.50

Assume $f, g \in \mathscr{L}^1(\mu)$. Then

1. $\|f\|_1 \geq 0$
2. $\|f\|_1 = 0 \iff f(x) = 0$ *for a.e.* $x \in X$
3. $\|cf\|_1 = |c|\|f\|_1$ *for all* $c \in \mathbb{R}$
4. $\|f + g\|_1 \leq \|f\|_1 + \|g\|_1$

Note: *by (3) and (4), $\mathscr{L}^1$ satisfies the properties of a vector space. However, by (2), $\|\cdot\|_1$ is not a norm.*

Theorem 2.51

Consider the measure space $(\mathbb{R}, \mathscr{L}, \lambda)$. Let $f \in \mathscr{L}^1(\lambda)$. Then $\forall \epsilon > 0$, $\exists g : \mathbb{R} \to \mathbb{R}$ such that g is continuous, $\{x \in \mathbb{R} : g(x) \neq 0\}$ is bounded and $\|f - g\|_1 < \epsilon$.

Definition 2.34

The **support** of a function $f : X \to [-\infty, \infty]$ is the closure of the non-zero domain

$$\overline{\{x \in X : f(x) \neq 0\}}$$

The set of all continuous function on $\mathbb{R}$ with compact support is denoted $C_c(\mathbb{R})$

☞ **Remark**

$C_c(\mathbb{R})$ is dense in $\mathscr{L}^1(\lambda)$

2.7 Product Measures

Definition 2.35

The **Cartesian Product** of X and Y is defined as

$$X \times Y = \{(x, y) : x \in X, y \in Y\}$$

Definition 2.36

Let X, Y be sets. A **rectangle** in $X \times Y$ is a set $A \times B$ with $A \subseteq X$, $B \subseteq Y$.

Definition 2.37

Given $(X, \mathscr{S}, \mu)$,$(Y, \mathscr{T}, \nu)$ The product $\mathscr{S} \otimes \mathscr{T}$ is defined to be the smallest σ-algebra containing all the rectangles generated by $\mathscr{S}$,$\mathscr{T}$:

$$\{A \times B : A \in \mathscr{S}, B \in \mathscr{T}\}$$

A **measurable rectangle** in $\mathscr{S} \otimes \mathscr{T}$ is a set of the form $A \times B$ where $A \in \mathscr{S}$ and $B \in \mathscr{T}$.

Definition 2.38

Let X,Y be sets. Let $E \subseteq X \times Y$. Then for $a \in X$, $b \in Y$ the **cross sections** $[E]_a$ and $[E]^b$ are defined as:

$$[E]_a = \{y \in Y : (a, y) \in E\}$$
$$[E]^b = \{x \in X : (x, b) \in E\}$$

Theorem 2.52

Let $(X, \mathscr{S})$, $(Y, \mathscr{T})$ be measurable spaces. If $E \in \mathscr{S} \otimes \mathscr{T}$, then $\forall a \in X$, $[E]_a \in \mathscr{T}$ and $\forall b \in Y$, $[E]^b \in \mathscr{S}$.

Definition 2.39

Let X,Y be sets. Let $f : X \times Y \to \mathbb{R}$. For $a \in X$, $b \in Y$, the **cross section functions** $[f]_a : Y \to \mathbb{R}$ and $[f]^b : X \to \mathbb{R}$ are defined to be

$$[f]_a(y) = f(a, y)$$
$$[f]^b(x) = f(b, x)$$

Note: $[f]_a$ is $\mathscr{T}$-measurable and $[f]^b$ is $\mathscr{S}$-measurable if f is $\mathscr{S} \otimes \mathscr{T}$-measurable.

Definition 2.40

A measure μ on $(X, \mathscr{S})$ is **finite** if $\mu(X) < \infty$.

Definition 2.41

μ is **σ-finite** if $\exists$ countably many sets $X_1, X_2, X_3, \ldots \in \mathscr{S}$ such that $\mu(X_k) < \infty$ for all $k \in \mathbb{N}$ and

$$X = \bigcup_{k=1}^{\infty} X_k$$

Definition 2.42

Let $(X, \mathscr{S}, \mu)$ and $(Y, \mathscr{T}, \nu)$ be measure spaces and $g : X \times Y :\to [-\infty, \infty]$.

$$\int_{X \times Y} g(x, y) d(\mu \times \nu) = \int_Y \int_X g(x, y) d\mu(x) d\nu(y)$$

Note that

$$\int_Y \int_X g(x, y) d\mu(x) d\nu(y) = \int_Y \Big(\int_X [g]^b d\mu(x) \Big) d\nu(y)$$

Theorem 2.53

The Riemann and Lebesgue integrals agree on $[a,b]$ if f is Riemann integrable on $[a,b]$:

$$\int_a^b f dx = \int_{[a,b]} f d\lambda$$

Theorem 2.54

Let $(X,\mathscr{S},\mu)$ and $(Y,\mathscr{T},\nu)$ be σ-finite. If $E \in \mathscr{S}\otimes\mathscr{T}$,

1. $x \mapsto \nu([E]_x)$ *is $\mathscr{S}$-measurable*
2. $y \mapsto \mu([E]^y)$ *is $\mathscr{T}$-measurable*

Definition 2.43

Let $(X,\mathscr{S},\mu)$ and $(Y,\mathscr{T},\nu)$ be σ-finite.

$$(\mu\times\nu)(E) = \int_X\int_Y \chi_E(x,y)d\nu(y)d\mu(x)$$

Remark measure of a rectangle

Let $A\in\mathscr{S}$, $B\in\mathscr{T}$

$$\begin{aligned}(\mu\times\nu)(A\times B) &= \int_X\int_Y \chi_{A\times B}(x,y)d\nu(y)d\mu(x)\\ &= \int_X\int_Y \chi_A\chi_B d\nu(y)d\mu(x)\\ &= \int_X \chi_x\nu(B)d\mu(x)\\ &= \mu(A)\nu(B)\end{aligned}$$

Theorem 2.55 Tonelli's

Let $(X,\mathscr{S},\mu)$ and $(Y,\mathscr{T},\nu)$ be measure spaces. Let $f: X\times Y\to[0,\infty]$ be $\mathscr{S}\otimes\mathscr{T}$ be $\mathscr{S}\otimes\mathscr{T}$-measurable. Then

1. $x\mapsto \int_Y f(x,y)d\nu(y)$ *is $\mathscr{S}$-measurable*
2. $y\mapsto \int_X f(x,y)d\mu(x)$ *is $\mathscr{T}$-measurable*
3. $\int_{X\times Y} fd(\mu\times\nu) = \int_X\int_Y f(x,y)d\nu(y)d\mu(x) = \int_Y\int_X f(x,y)d\mu(x)d\mu(y)$

Theorem 2.56

If $\{x_{j,k}\}_{j\in\mathbb{N},k\in\mathbb{N}}$ are $x_{j,k}\ge 0$ for all j,k, then

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} x_{j,k} = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} x_{j,k}$$

Theorem 2.57

Suppose $(X, \mathscr{S}, \mu)$ and $(Y, \mathscr{T}, \nu)$ are σ-finite and $f : X \times Y \to [-\infty, \infty]$ is $\mathscr{S} \otimes \mathscr{T}$-measurable. Assume $f \in \mathscr{L}^1(\mu \times \nu)$. Then

1. $\int_Y |f(x, y)| d\nu(y) < \infty$ *for a.e.* $x \in X$
2. $\int_X |f(x, y)| d\mu(x) < \infty$ *for a.e.* $y \in Y$
3. $x \mapsto \int_Y f(x, y) d\nu(y)$ *is* $\mathscr{S}$*-measurable*
4. $y \mapsto \int_X f(x, y) d\mu(x)$ *is* $\mathscr{T}$*-measurable*
5. $\int_{X \times Y} f d(\mu \times \nu) = \int_X \int_Y f(x, y) d\nu(y) d\mu(x) = \int_Y \int_X f(x, y) d\mu(x) d\mu(y)$

2.8 Lebesgue Integration on $\mathbb{R}^n$

Definition 2.44

If $x \in \mathbb{R}^n$, $\delta > 0$, we define

$$B(x, \delta) = \{y \in \mathbb{R}^n : \|y - x\|_\infty < \delta\}$$

to be the **open cube**.

Definition 2.45

A set $G \subseteq \mathbb{R}^n$ is **open** if $\forall x \in G$ there exists $\delta > 0$ such that $B(x, \delta) \subseteq G$.

☞ **Remark**

$$B(x, \delta) \in \mathbb{R}^m \times B(y, \delta) \in \mathbb{R}^n = B((x, y), \delta) \in \mathbb{R}^{m+n}$$

☞ **Remark**

Let $G_1 \subseteq \mathbb{R}^m$ open and $G_2 \subseteq \mathbb{R}^n$ open. Then

$$G_1 \times G_2 = \mathbb{R}^{m+n}$$

Definition 2.46

Borel Set in $\mathbb{R}^n$ is an element of the smallest σ-algebra on $\mathbb{R}^n$ which contains all open subsets of $\mathbb{R}^n$. Denote this σ-algebra $\mathbb{B}_n$

Theorem 2.58

$G \subseteq \mathbb{R}^n$ is open $\iff$ G is a countable union of open cubes in $\mathbb{R}^n$

☞ **Remark**

$\mathbb{B}_n$ is the smallest σ-algebra containing all the open cubes in $\mathbb{R}^n$

Theorem 2.59

$$\mathbb{B}^{m+n} = \mathbb{B}^n \otimes \mathbb{B}^m$$

Definition 2.47

$$(\mathbb{R}^2, \mathbb{B}_2, \lambda_2) = (\mathbb{R}, \mathbb{B}, \lambda) \times (\mathbb{R}, \mathbb{B}, \lambda)$$

Lebesgue Measure on $\mathbb{R}^n$ is denoted λ_n and defined as

$$\lambda_n = \lambda_{n-1} \times \lambda_1$$

☞ **Remark**

Let $(\mathbb{R}^n, \mathbb{B}_n, \lambda_n)$ be a measure space. Then

$$\mathbb{B}_n = \mathbb{B}_{n-1} \times \mathbb{B}_1$$

So for $E \in \mathbb{B}^n$,

$$\begin{aligned}
\lambda_n(E) &= \int_{\mathbb{R}^n} \chi_E(x) d\lambda_n(x) \\
&= \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \chi_E(x_1, x_2) d\lambda(x_1) d\lambda_{n-1}(x_2) \\
&\vdots \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \dots \int_{\mathbb{R}} \chi_E(x_1, x_2, \dots, x_n) d\lambda(x_1) d\lambda(x_2) \dots d\lambda(x_n)
\end{aligned}$$

Chapter 3 Hilbert Spaces (MTH 513)

3.1 Banach Spaces

3.1.1 Integration on $\mathbb{C}$

Definition 3.1

The set of all complex numbers $\mathbb{C}$ is given by

$$\mathbb{C} = \{z = a + bi : a, b \in \mathbb{R}, i^2 = -1\}$$

Definition 3.2

Given $z \in \mathbb{C}$ where $z = a + bi$, the **Real** and **Imaginary** parts of z are given by

$$\Re(z) = a$$
$$\Im(z) = b$$

Note that both $\Re(z), \Im(z) \mapsto \mathbb{R}$ and $z = \Re(z) + \Im(z)i$

Definition 3.3

The **modulus** of $z \in \mathbb{C}$ is given by

$$|z| = (a^2 + b^2)^{1/2}$$

Definition 3.4

The **complex conjugate** of $z \in \mathbb{C}$ is given by

$$\bar{z} = \Re(z) - \Im(z)i$$

Theorem 3.1

Properties of complex conjugates:

- *products:*

$$z\bar{z} = |z|^2$$

- *sums and differences*

$$z + \bar{z} = 2\Re(z)$$
$$z - \bar{z} = 2\Im(z)i$$

- *multiplicativity and additivity*

$$\overline{w + z} = \bar{w} + \bar{z}$$
$$\overline{wz} = \bar{w}\bar{z}$$

- *conjugates of conjugates*

$$\bar{\bar{z}} = z$$

✿ *absolute value*

$$|\bar{z}| = |z|$$

✿ *integral of conjugate function*

$$\int \bar{f} d\mu = \overline{\int f du}$$

Definition 3.5

Let $(X, \mathscr{S})$ be a measurable space. $f : X \to \mathbb{C}$ is $\mathscr{S}$-measurable if both $\Re(f) : X \to \mathbb{R}$ and $\Im(f) : X \to \mathbb{R}$ are $\mathscr{S}$-measurable.

Theorem 3.2

Suppose $(X, \mathscr{S})$ is a measurable space, $f : X \to \mathbb{C}$ is $\mathscr{S}$-measurable, and $0 < p < \infty$. Then $|f|^p$ is also $\mathscr{S}$-measurable.

Definition 3.6

Suppose $(X, \mathscr{S}, /mu)$ is a measure space and $f : X \to \mathbb{C}$ is $\mathscr{S}$-measurable. Assume $f \in \mathscr{L}^1(\mu)$. We define

$$\int_X f d\mu = \int_X \Re(f) d\mu + i \int_X \Im(f) d\mu$$

☞ **Remark**

If $f, g : X \to \mathbb{C}$ are $\mathscr{S}$-measurable and $f, g \in \mathscr{L}^1(\mu)$, then

1. $\displaystyle\int (f+g) d\mu = \int f d\mu + \int g d\mu$
2. $\displaystyle\int \alpha f d\mu = \alpha \int f d\mu, \forall \alpha \in \mathbb{C}$

Theorem 3.3

Suppose $(X, \mathscr{S}, \mu)$ is a measure space and $f : X \to \mathbb{C}$ is $\mathscr{S}$-measurable. Assume $f \in \mathscr{L}^1(\mu)$. Then

$$\left| \int_X f d\mu \right| \leq \int_x |f| d\mu$$

3.1.2 Bounded Linear Operators

Definition 3.7

For notation, we let the field $\mathbb{F}$ denote either $\mathbb{R}$ or $\mathbb{C}$

Definition 3.8

Let V, W be vector spaces. A function $T : V \to W$ is a **linear operator** or **linear map** if

1. $T(f+g) = Tf + Tg, \forall f, g \in V$
2. $T(\alpha f) = \alpha T f, \forall \alpha \in \mathbb{F}$ and $\forall f \in V$

Definition 3.9

Let $(V, \|\cdot\|_V), (W, \|\cdot\|_W)$ be NLS and $T : V \to W$. Recall that the **Operator Norm** on T is given by

$$\begin{aligned} |||T||| &= \sup_{\|f\|_V \leq 1} \{\|Tf\|_W\} \\ &= \sup_{\|f\|_V = 1} \{\|Tf\|_W\} \\ &= \sup\{\frac{\|Tf\|_W}{\|f\|_V} : \|f\|_V \neq 0\} \end{aligned}$$

If $|||T||| < \infty$, then T is a **bounded linear operator**. The set of all bounded linear operators $T : V \to W$ is denoted

$$B(V, W)$$

and we sometimes write

$$|||T||| = \|T\|_{B(V,W)}$$

☞ **Remark**

$B(V, W)$ is a vector space. Moreover, $|||T|||$ is a norm on $B(V, W)$ so

$$\left(B(V, W), \|T\|_{B(V,W)}\right)$$

is a NLS.

Theorem 3.4

Suppose $(V, \|\cdot\|_V), (W, \|\cdot\|_W)$ are NLSs and $T : V \to W$ is a bounded linear operator. T is not a bounded function.

Proof. Let $\alpha \in \mathbb{F}$ and $f \in V$ such that $Tf \neq 0$.

$$\begin{aligned} \|T(\alpha f)\|_W &= \|\alpha Tf\|_W \\ &= |\alpha|\|Tf\|_W \to \infty \\ &\text{as } |\alpha| \to \infty \end{aligned}$$

So $\nexists R > 0$ such that $\|Tf\|_W \leq R, \forall f \in V$. Therefore T is not a bounded function. ■

Theorem 3.5

Let $C[a, b]$ be the set of all continuous functions on $[a, b]$ and let $C^1[a, b]$ be the set of all functions with continuous first order derivatives on $[a, b]$. If we define the norms

$$\begin{aligned} \|f\|_{C^1[a,b]} &= \|f\|_\infty + \|f'\|_\infty \\ &\textit{and} \\ \|f\|_{C[a,b]} &= \|f\|_\infty \end{aligned}$$

then $T : \left(C^1[a, b], \|f\|_{C^1[a,b]}\right) \to \left(C[a, b], \|f\|_{C[a,b]}\right)$, where $Tf = f'$, is a bounded linear operator.

Theorem 3.6

Suppose $(V, \|\cdot\|_V)$, $(W, \|\cdot\|_W)$ are NLSs with $V \neq \{0\}$ and $T : V \to W$ is a linear map. Then

$$|||T||| = \sup_{\|f\|_V}\{\|Tf\|_W\} = \sup_{f \neq 0}\{\frac{\|Tf\|_W}{\|f\|_V}\}$$

So we can write the inequality

$$\|Tf\|_W \leq |||T||| \|f\|_V$$

This shows that $|||T|||$ can be thought of as the smallest value such that the above inequality holds.

Theorem 3.7

If $(W, \|\cdot\|_W)$ is a Banach Space and $(V, \|\cdot\|_V)$ is any NLS (not necessarily complete) then $B(V, W)$ is also a Banach Space.

Theorem 3.8

Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be NLS. A linear map $T : V \to W$ is continuous if and only if it is bounded.

3.2 Baire Category Theorem

Definition 3.10

Let $U \subseteq V$ where V is a metric space. Recall that the **interior** of U is

$$\text{int}(U) = \{f \in U : \exists r > 0 \text{ s.t. } B_r(f) \subseteq U\}$$

☞ **Remark**

$\text{int}(U)$ is open in V.

Definition 3.11

Recall that U is **dense** in V:

$\iff \overline{U} = V$

$\iff f$ is a limit point of U for all $f \in V$.

$\iff \forall f \in V$ and $\forall r > 0$, $B_r(f) \cap U \neq \emptyset$

Definition 3.12

A subset $E \subseteq V$ is **nowhere dense** in V

$\iff \underline{V - \overline{U}}$ is dense in V

$\iff \overline{V - \bar{E}} = V$

$\iff \text{int}(\bar{E}) = \emptyset$

Example 3.1

✿ $\mathbb{Z}$ is nowhere dense in $\mathbb{R}$ ($\mathbb{R} - \bar{\mathbb{Z}} = \mathbb{R} - \mathbb{Z}$

✿ A line is nowhere dense in $\mathbb{R}^2$

✿ A line or a plane is nowhere dense in $\mathbb{R}^3$

Theorem 3.9 Baire Category Theorem

(a) A complete metric space is not the countable union of closed subsets with empty interiors.

(b) The countable intersection of dense, open subsets of a complete metric space is non-empty.

Remark

(a) says that a complete metric space is not the countable union of nowhere dense sets. So, for example, we cannot represent $\mathbb{R}^3$ as the countable union of planes.
(a) also implies that, if X is a complete metric space and the countable union of closed sets G, then at least one G is non-empty, so that G contains a non-empty open set.

? Uniform Boundedness Principle

Theorem 3.10

Assume V is a Banach Space and W is any NLS. Let $\mathscr{A}$ be the set of bounded linear maps from $V \to W$ such that

$$\sup\{\|Tf\|_W : T \in \mathscr{A}\} < \infty$$

Then $\sup\{|||T||| : T \in \mathscr{A}\} < \infty$ (i.e. the Ts are uniformly bounded).

3.2.1 Open Mapping Theorem

Theorem 3.11

Let V, W be Banach Spaces and T be a bounded linear surjection. If G is open in V, then $T(G)$ is open in W.

Corollary 3.1

Let V, W be Banach Spaces and T be a bounded linear bijection, then T^{-1} is a bounded linear map. (i.e. $T^{-1} : W \to V$ is continuous).

3.3 L^p Spaces

3.3.1 $\mathscr{L}^p$ Spaces

Definition 3.13

Let $(X, \mathscr{S}, \mu)$ be a measure space, fix $p \in (0, \infty)$ and let $f : X \to \mathbb{F}$ be $\mathscr{S}$-measurable. Then the **p-norm** of f is

$$\|f\|_p = \left(\int_X |f|^p d\mu \right)^{1/p}$$

Definition 3.14

The **essential supremum** of f is

$$\|f\|_\infty = \inf\{t > 0 : |f(x)| \leq t \text{ a.e.}\}$$

In other words, the smallest upper bound of the function on all sets, except those of measure 0.

☞ **Remark** Motivation for $1/p$ in **p-norm** definition
Consider $0 < p < \infty$ and $\alpha \in \mathbb{F}$. Take some $f : X \to \mathbb{F}$. Then

$$\begin{aligned}\|\alpha f\|_p &= \Big(\int_X |\alpha f|^p d\mu\Big)^{1/p} \\ &= \Big(\int_X |\alpha|^p |f|^p d\mu\Big)^{1/p} \\ &= |\alpha|^p \Big(\int_X |f|^p d\mu\Big)^{1/p} \\ &= |\alpha| \|f\|_p\end{aligned}$$

But without the exponent $1/p$, we get $\|\alpha f\|_p = |\alpha|^p \|f\|_p$, which violates the definition of a norm! So we really do need the $1/p$ to make $\|f\|_p$ a norm on f.

Definition 3.15

Let $(X, \mathscr{S}, \mu)$ be a measure space and $0 < p < \infty$. **Lebesgue Space**, $\mathscr{L}^p(\mu)$ is the set of all $\mathscr{S}$-measurable functions $f : X \to \mathbb{F}$ such that

$$\|f\|_p < \infty$$

Intuition for $\| \cdot \|_p$

☞ **Remark**

1. **What does $\|f\|_p$ tell us about f locally?**
Say the function $f : X \to \mathbb{F}$ blows up (i.e. grows unbounded) near some $x \in X$. Then f is not Riemann integrable, but f may be integrable in some $\mathscr{L}^p$ space. For example, consider the function

$$f(x) = \frac{1}{|x|}$$

Where $f : B(0,1) \to \mathbb{R}$ and $B(0,1) \in \mathbb{R}^2$. Note that as $x \to 0$, $f \to \infty$. However we can show that

$$\|f\|_1 = \int_{B(0,1)} |f| d\lambda$$

(by change of coordinates)

$$\begin{aligned} &= 2\pi k \int_0^1 r \frac{1}{r} dr \\ &= 2\pi k < \infty \end{aligned}$$

Now consider the following:

$$\|f\|_{3/2}^{3/2} = \int_{B(0,1)} \frac{1}{|x|^{3/2}} d\lambda$$

(by change of coordinates)

$$= 2\pi k \int_0^1 r \frac{1}{r^{3/2}} dr < \infty$$

In fact, we can show that $f \in \mathscr{L}^p$ for all $p < 2$, but $f \notin \mathscr{L}^p$ for $p \geq 2$

Take away: Given $f \in \mathscr{L}^p$, the larger p is, the slower the localized function grows unbounded. So if $f \in \mathscr{L}^1$, then the function grows unbounded rapidly, but if $f \in \mathscr{L}^{100}$, the function grows unbounded much slower!

2. **What does $\|f\|_p$ tell us about how the function decays as $|x| \to \infty$?**

   Consider some $p \in [1, \infty)$. In order for $f \in \mathscr{L}^p$ to hold, we need the function to decay (i.e. approach 0) as $|x| \to \infty$. But when we take

$$\int_X |f|^p d\mu$$

   raising the function the p-th power results in even faster decay at $x = \infty$.

   Take Away: Given $f \in \mathscr{L}^p(\mathbb{R}^n)$, then the smaller p is, the faster f decays at ∞ because it needs less help from the power of p to make the norm finite!

Definition 3.16

Let $1 \leq p \leq \infty$. Then the **dual exponent** of p, denoted q (or sometimes p') is the number that satisfies

$$\frac{1}{p} + \frac{1}{q} = 1$$

Note: for $p = \infty$, $q = 1$.

Theorem 3.12 Young's Inequality

Let $p \in (0, \infty)$ and q be the dual exponent of p. Then $\forall a, b \geq 0$,

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

Theorem 3.13 H"older's Inequality

Let $p \in [0, \infty]$, $(X, \mathscr{S}, \mu)$ be a measure space, and $f, g : X \to \mathbb{F}$ be $\mathscr{S}$-measurable functions. Then

$$\|fg\|_1 \leq \|f\|_p \|g\|_q$$

$$\int_X |fg| d\mu \leq \Big[\int_X |f|^p d\mu\Big]^{1/p} \Big[\int_X |g|^q d\mu\Big]^{1/q}$$

Theorem 3.14

*Let $(X, \mathscr{S}, \mu)$ be a finite measure space ($\mu(X) < \infty$) and $0 < p < s < \infty$ (note: s **not necessarily** the dual exponent of p). Then*

$$\|f\|_p \leq \mu(X)^{\frac{s-p}{p}} \|f\|_s$$

This implies that when $\mu(X) < \infty$ and $p < s$,

$$f \in \mathscr{L}^s \Rightarrow f \in \mathscr{L}^p$$

$$\text{so,}$$

$$\mathscr{L}^s \subseteq \mathscr{L}^p$$

Furthermore, consider the case where $0 < p < s = \infty$. Then

$$\|f\|_p = \Big[\int_X |f|^p d\mu\Big]^{1/p} \leq \Big[\int_X \|f\|_\infty d\mu\Big]^{1/p}$$

Recall that $\|f\|_\infty = \inf_{x \in X}\{M : |f(x)| \leq M \text{ a.e.}\}$. So,

$$\Big[\int_X \|f\|_\infty d\mu\Big]^{1/p} = \|f\|_\infty\Big(\int_X 1 d\mu\Big)^{1/p} = \|f\|_\infty \mu(X)^{1/p}.$$

Therefore, if $\mu(X) < \infty$,

$$\mathscr{L}^\infty(\mu) \subseteq \mathscr{L}^p(\mu), \ \forall p < \infty$$

3.3.2 L^p Spaces

Definition 3.17

Let $(X, \mathscr{S}, \mu)$ be a measure space and $0 < p \leq \infty$.

(i) $Z(\mu)$ is the set of all $\mathscr{S}$-measurable functions $X \mapsto \mathbb{F}$ which are equal almost everywhere on X

(ii) For $f \in L^p(\mu)$, let $\tilde{f}$ denote the subset of $\mathscr{L}^p(\mu)$,

$$\tilde{f} = \{f + z : z \in Z(\mu)\} = f + Z(\mu)$$

☞ Remark

Let $f_1, f_2 \in \tilde{f}$. Then $\exists z_1, z_2 \in Z(\mu)$ such that

$$f_1 = f + z_1$$
$$f_2 = f + z_2$$
$$f_1 - f_2 = z_1 - z_2 = 0 \text{ a.e.}$$

So $f_1 = f_2$ a.e.

☞ Remark

Suppose $\tilde{f} = \tilde{g}$. Then $f + Z(\mu) = g + Z(\mu)$, so $f = f + 0 \in g + Z(\mu) \Rightarrow \exists z \in Z(\mu)$ such that $f = g + z$. Therefore $f = g$ a.e.

Definition 3.18 L^p

Let $0 < p \leq \infty$

$$L^p(\mu) = \{\tilde{f} : f \in \mathscr{L}^p(\mu)\}$$

Definition 3.19

Let $0 < p \leq \infty$. We define $\|\cdot\|_p$ on $L^p(\mu)$ by

$$\|\tilde{f}\|_p = \|f\|_p, \ \forall f \in \mathscr{L}^p(\mu)$$

If we restrict $p \in [1, \infty]$, then $\|\cdot\|_p$ is a norm on $L^p(\mu)$.

☞ **Remark**
Consider $\tilde{f} = \tilde{g}$. Then $\|\tilde{f}\|_p = \|\tilde{g}\|_p$.

3.3.3 Dual of L^p

Theorem 3.15

Let $(X, \mathscr{S}, \mu)$ be a measure space, $p \in [1, \infty)$, q be the dual exponent of p, and $f \in L^p(\mu)$. Then

$$\|f\|_p = \sup\left\{\left|\int_X fhd\mu\right| : h \in \mathscr{L}^q(\mu), \ \|h\|_q \leq 1\right\}$$

☞ **Remark**
The above holds for $p = \infty \iff \mu$ is σ-finite.

☞ **Remark**
Recall that the **dual space** of an NLS X, denoted X^* is defined as the the set of all bounded linear functionals on X

$$X^* = \{f : f : X \to \mathbb{F}\}$$

Theorem 3.16

Let $(X, \mathscr{S}, \mu)$ be a measure space, $1 < p \leq \infty$, and q be the dual exponent of p. For $h \in L^q(\mu)$, define $\phi_h : L^p(\mu) \to \mathbb{F}$ by

$$\phi_h(f) = \int_X fhd\mu$$

Note the following are true:

(i) $h \mapsto \phi_h$ *is 1:1, linear, and maps* $L^q(h)$ *into* $(L^p(\mu))^*$

(ii) $|||\phi_h||| = \|h\|_q, \ \forall h \in L^q(\mu)$

In fact, since $L^q(\mu)$ has a 1:1 correspondence with $(L^p(\mu))^*$, we can show that $L^q(\mu) = (L^p(\mu))^*$.

Theorem 3.17

Let $T(h) = \phi_h$. We can show that T is linear and $h \xrightarrow{T} \phi_h$ is 1:1, so $(L^p(\mu), \|\cdot\|_p)$ is an NLS.

Furthermore, for $1 \leq p \leq \infty$, $(L^p(\mu), \|\cdot\|_p)$ is complete, therefore

$$(L^p(\mu), \|\cdot\|_p) \text{ is a Banach Space.}$$

☞ Remark

Let $(X, \mathscr{S}, \mu)$ be a measure space. Recall that for $f \in \mathscr{L}^p(\mu)$, $\forall \epsilon > 0$, $\exists \phi \in \mathscr{L}^1(\mu)$, where ϕ is a simple function, such that

$$\|f - \phi\|_1 < \epsilon$$

Theorem 3.18

The above also holds for $L^p(\mu)$, $\forall 1 \leq p \leq \infty$.

Theorem 3.19

Let $f \in L^\infty(\mu)$ and $\epsilon > 0$. There exists $\phi \in L^\infty(\mu)$ where ϕ is simple and

$$\|f - g\|_\infty$$

Theorem 3.20

Suppose $f \in L^p(\mathbb{R})$ and $0 < p < \infty$. Then $\forall \epsilon > 0$ $\exists$ a step function $g \in L^p(\mathbb{R})$ such that

$$\|f - g\|_p < \epsilon$$

3.4 Hilbert Spaces

3.4.1 Inner Product Spaces

Definition 3.20

Let V be a vector space over $\mathbb{F}$. An **inner product** on V is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$ such that

(i) $\langle f, f \rangle \in [0, \infty)$

(ii) $\langle f, f \rangle = 0 \iff f = 0$

(iii) $\langle f + g, h \rangle = \langle f, h \rangle + \langle g, h \rangle$

(iv) $\langle \alpha f, g \rangle = \alpha \langle f, g \rangle$

(v) $\langle f, g \rangle = \overline{\langle g, f \rangle}$

Definition 3.21

An **Inner Product Space** (IPS) is a vector space with an inner product.

☞ Remark

Let $f, g \in L^2(\mu)$ and define

$$\langle f, g \rangle = \int_X f\bar{g}\,d\mu$$

Then $L^2(\mu)$ is an IPS.

Theorem 3.21

Suppose V is an IPS. Then the following properties also hold:

(a) $\langle 0, g\rangle = \langle g, 0\rangle$

(b) $\langle f, g+h\rangle = \langle f, g\rangle + \langle f, h\rangle$

(c) $\langle f, \alpha g\rangle = \overline{\alpha}\langle f, g\rangle$

Definition 3.22

Let V be an IPS. We can induce a norm on V ny

$$\|f\| = \sqrt{\langle f, f\rangle}$$

Theorem 3.22 Properties of $\|\cdot\|$

1. $\|f\| = \sqrt{\langle f, f\rangle} \geq 0 \ \forall f \in V$
2. $\|f\| = \sqrt{\langle f, f\rangle} = 0 \iff f = 0$
3. $\|\alpha f\| = \alpha\|f\|$

Definition 3.23

Let V be an IPS and $x, y \in V$. x, y are **orthogonal** if $\langle x, y\rangle = 0$. We write this $x \perp y$.

Theorem 3.23 Pythagorean Theorem

Assume V is an IPS and $f, g \in V$ with $\langle f, g\rangle = 0$. Then

$$\|f+g\|^2 = \|f\|^2 + \|g\|^2$$

Theorem 3.24 Cauchy Schwarz

Let V be an IPS and $f, g \in V$. Then

$$|\langle f, g\rangle| \leq \|f\|\|g\|$$

Theorem 3.25

Let V be an IPS and $f, g \in V$. Then

$$\|f+g\| \leq \|f\| + \|g\|$$

3.4.2 Angles in an IPS

Definition 3.24

Define the angle θ between f, g in an IPS by

$$\cos\theta = \frac{\langle f, g\rangle}{\|f\|\|g\|} \in [-1, 1]$$

Theorem 3.26 Law of Cosines

Let $a = \|f\|$, $b = \|g\|$, $c = \|f - g\|$.

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

Theorem 3.27 Parallelogram Equality

Let V be an IPS and $f, g \in V$. Then

$$\|f+g\|^2 + \|f-g\|^2 = 2\|f\|^2 + 2\|g\|^2$$

3.4.3 Orthogonality

Definition 3.25 Hilbert Space

A **Hilbert Space** is any IPS that is complete under the norm induced by the inner product.

Definition 3.26

Suppose U is a non-empty subset of an NVS V and $f \in V$. The distance from f to U is

$$\text{distance}(f, U) = \inf\{\|f - g\| : g \in U\}$$

☞ **Remark**

If U is open, then

$$\inf\{\|f-g\| : g \in U\} \neq \min\{\|f-g\| : g \in U\}$$

☞ **Remark**

$$\text{distance}(f, U) = 0 \iff f \in \overline{U}$$

Definition 3.27

Suppose V is a vector space and $U \subseteq V$. U is **convex** if $\forall f, g, \in U$ and $t \in [0, 1]$

$$(1-t)f + tg \in U$$

☞ **Remark**

Every vector space is convex since it is closed under linear combination

3.4.4 Orthogonal Projection

Theorem 3.28

let V be a Hilbert Space, $U \subseteq V$ be closed, convex, and non-empty, and $f \in V$. Then $\exists ! g \in U$ such that

$$\|f - g\| = \textit{distance}(f, U)$$

Definition 3.28

Suppose V is a Hilbert Space and $U \in V$ is a closed, non-empty, convex subset of V. The **orthogonal**

projection of V onto U is the function

$$P_U : V \to U$$

where $P_U f$ is the unique element of U that best approximates $f \in V$.

☞ **Remark**

(a) $P_U f = 0 \iff f \in U$

(b) $P_U \circ P_U = P_U^2 = P_U$

Theorem 3.29

Suppose U is a closed subspace of a Hilbert Space V. For $f \in V$:

(a) $f - P_U f \perp g \ \forall g \in U$

(b) *If $h \in U$ and $f - h \perp g \ \forall g \in U$, $h = P_U h$*

(c) *$P_U : V \to U$ is a linear map*

(d) *$\forall f \in V$, $\|P_U f\| \le \|f\|$ and $\|P_U f\| = \|f\| \iff f \in U$*

Example 3.2

Recall

$$\ell^2 = \{a = (a_1, a_2, \dots),\ a_j \in \mathbb{F},\ \sum_{j=1}^{\infty} |a_j|^2 < \infty\}$$

So $\forall x, y \in \ell^2$,

$$\langle x, y \rangle = \sum_{j=1}^{\infty} x_j \overline{y_j}$$

Consider the subset

$$U = \{a \in \ell^2 : a = (a_1, 0, a_3, 0, a_5, 0, \dots)\}$$

So, given $x \in \ell^2$,

$$P_U x = (x_1, 0, x_3, 0, x_5, 0, \dots)$$

Then we have

$$x - P_U x = (0, x_2, 0, x_4, 0, x_6 \dots)$$

so

$$\langle x, x - P_U \rangle = \sum_{j=1}^{\infty} (x_j)\overline{(x_j - P_U x_j)} = 0$$

$$\Rightarrow x \perp x - P_U x$$

Definition 3.29

Suppose U is a subset of an IPS V. The orthogonal complement of U in V is

$$U^\perp = \{h \in U : \langle h, g \rangle = 0 \ \forall g \in U\}$$

Example 3.3

Let V be an IPS and $U \subseteq V$
If $U = V \Rightarrow U^{\perp} = \{0\}$.
Suppose $U = B(0,1) = \{g \in V : \|g\| = 1\}$. Then $U^{\perp} = \{0\}$ because for $x \in U^{\perp}$,

$$\begin{aligned} \langle x, y\rangle &= 0 \ \forall y \in U^{\perp} \\ h \in V^{\perp} \Rightarrow \langle x, h\rangle &= \langle x, \|h\| \frac{h}{\|h\|}\rangle \\ &= \|h\| \langle x, \frac{h}{\|h\|}\rangle = 0 \end{aligned}$$

Since $\frac{h}{\|h\|} \in B(0,1)$.

3.4.5 Properties of Orthogonal Projections

Theorem 3.30

Let V be an IPS and $U \subseteq V$. Then

(a) $U^{\perp}$ *is a closed subspace of* V

(b) $U \cap U^{\perp} = \{0\}$ *if* $0 \in U$*, otherwise* $\varnothing$*. So* $U \cap U^{\perp} subseteq \{0\}$

(c) *If* $W \subset U$, $U^{\perp} \subseteq W^{\perp}$

(d) $\overline{U^{\perp}} = U^{\perp}$

(e) $U \subseteq (U^{\perp})^{\perp}$

Theorem 3.31 Orthogonal Decomposition

Let U be a closed subspace of a Hilbert Space V. Then any $f \in V$ can be written as

$$f = g + h$$

where $g \in U$ and $h \in U^{\perp}$

Theorem 3.32 Range and Null Space of P_U

Suppose U is a closed subspace of a Hilbert Space V. Then the following are true:

(a) $Range(P_U) = U$, $Null(P_U) = U^{\perp}$

(b) $Range(P_{U^{\perp}}) = U^{\perp}$, $Null(P_{U^{\perp}}) = U$

(c) $P_{U^{\perp}} = \mathbf{I} - P_U$ *where* $\mathbf{I}$ *is the identity function*

Example 3.4

Let

$$U = \{f \in L^2(\mathbb{R}) : f(x) = 0 \text{ a.e. } x < 0\}$$

We can show that U is a closed subspace of L^2. So

$$U^{\perp} = \{f \in L^2(\mathbb{R}) : f(x) = 0 \text{ a.e. } x \geq 0\}$$

Theorem 3.33 Riesz Representation Theorem

Let V be a Hilbert Space. Suppose $\phi \in V^*$. Then $\exists! h \in V$ such that $\phi(g) = \langle g, h \rangle \ \forall g \in V$, so ϕ h and $\|\phi\| = \|h\|$

3.4.6 Orthonormal Bases

Definition 3.30

Let V be an NLS and consider $\{e_k\}_{k\in\Gamma} \subset V$ where

$$\Gamma = \{1, 2, 3, \dots, n\}$$

$$or$$

$$\Gamma = \mathbb{N}$$

A family $\{e_k\}_{k\in\Gamma}$ in an IPS is an **orthonormal family** if

$$\langle e_j, e_k \rangle = \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases}$$

Example 3.5 Example: $\mathbb{R}^n$

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \ e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \ \dots \ e_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$\{e_k\}_{k\in\Gamma}$ is an orthonormal family.

Example 3.6

Example: $\ell^2(\mathbb{F})$]

$$e_k = (0, 0, 0, \dots, 0, 1, 0, \dots, 0, 0, 0)$$

where the k-th element is 1. $\{e_k\}_{k\in\Gamma}$ is an orthonormal family.

3.4.7 Basis of a Hilbert Space

Definition 3.31

Recall: A metric space is **seperable** if it has a countable, dense subset.

Theorem 3.34

Every seperable Hilbert Space has a countable orthonormal basis. Moreover, if V is an infinite dimensional Hilbert Space, then there exists a countable orthonormal family $\{e_k\}_{k\in\mathbb{N}}$ such that $\forall f \in V$,

$\exists!$ sequence $\{c_k\}_{k\in\mathbb{N}}$, $c_k \in \mathbb{F}$, such that

$$\|f - \sum_{k=1}^{N} c_k e_k\| \to 0$$

as $N \to \infty$. So $f = \sum_{k=1}^{\infty} c_k e_k$

Example 3.7

Example: $\mathbb{R}^2$] Let $\{q_1, q_2\}$ be an orthonormal family in $\mathbb{R}^2$ and $f \in \mathbb{R}^2$. Then

$$f = \langle f, q_1\rangle + \langle f, q_2\rangle q_2$$

Example 3.8 Example: $\mathbb{R}^2$

Let $\{q_k\}_{1\leq k\leq n}$ be an orthonormal family in $\mathbb{R}^n$ and $f \in \mathbb{R}^n$. Then $\exists!$ collection $\{c_1, c_2, \ldots, c_n\}$ such that

$$f = \sum_{k=1}^{n} c_k q_k$$

What are the c_ks?

$$\begin{aligned}
\langle f, q_k\rangle &= \langle \sum_{j=1}^{n} c_j q_j, q_k\rangle \\
&= \sum_{j=1}^{n} c_j \langle q_j, q_k\rangle \\
&= c_k(1) \\
&= c_k
\end{aligned}$$

so $f = \sum_{k=1}^{\infty} \langle f, q_k\rangle q_k$.

Example 3.9 Example: Infinite Dimensional Hilbert Space

Let V be an infinite dimensional, seperable, Hilbert Space. Let $\{e_k\}_{k\in\mathbb{N}}$ be an orthonormal family in V. Moreover, assume $\{e_k\}_{k\in\mathbb{N}}$ is an orthonormal basis for V. So, given $f \in V$:

$$f = \sum_{k=1}^{\infty} c_k e_k$$

for some $\{c_k\}_{k\in\mathbb{N}}$. It can be shown that

$$\{c_k\}_{k\in\mathbb{N}} = \{\langle f, e_1\rangle, \langle f, e_2\rangle, \langle f, e_3\rangle, \ldots, \langle f, e_n\rangle\}$$

3.4.8 Bessel's Inequality

Theorem 3.35

Let $\{e_k\}$ be an orthonormal family in a Hilbert Space V. then $\forall f \in V$ and $\forall n \in \mathbb{N}$,

$$\|f\|^2 \geq \sum_{j=1}^{n} |\langle f, e_j \rangle|^2$$

Furthermore,

$$\|f\|^2 \geq \sum_{j=1}^{\infty} |\langle f, e_j \rangle|^2$$

.

3.4.9 Parceval's Identity

Theorem 3.36

Suppose $\{e_k\}_{k\in\mathbb{N}}$ is an orthonormal ***basis*** *for a seperable Hilbert Space V. let $f \in V$. Then*

$$\|f\|^2 = \sum_{j=1}^{n} |\langle f, e_j \rangle|^2$$

3.4.10 Linear Maps on Hilbert Spaces

Definition 3.32

Let V, W be Hilbert Spaces and $T : V \to W$ be a bounded linear map. The **adjoint** of T, $T^* : W \to V$ is defined as

$$\langle Tf, g \rangle = \langle f, T^* g \rangle$$

$\forall f \in V, \forall g \in W$.

☞ **Remark** Intuition:

Fix $g \in W$. Consider a linear functional $\phi_g^T \in V$ defined by

$$\phi_g^T(f) = \langle Tf, g \rangle$$

(Note: since T is linear and $\langle \cdot, \cdot \rangle$ is linear in the first slot, ϕ_g^T is linear).

$$|\phi_g^T(f)| = |\langle Tf, g \rangle| \leq \|Tf\|\|g\| \leq \|T\|\|f\|\|g\|$$

so $\|\phi_g^T\| \leq \|T\|\|g\|$, which implies $\phi_g^T \in V^*$.
Now, by the Riesz Representation Theorem, $\exists! h \in V$ such that

$$\phi_g^T(f) = \langle f, h \rangle$$

$\forall f \in V$. So for $g \in W$, set $T^* g = h$ where h is the unique element of V given by the RRT.

Example: Let $(X, \mathscr{S}, \mu)$ be a measure space and $h \in L^\infty(\mu)$. Define $M_h : L^2(\mu) \to L^2(\mu)$ by

$$M_h(f) = fh$$

$\forall f \in L^2(\mu)$. Then

$$\|M_h f\|_2 \leq \|fh\|_2 \leq \|f\|_2 \|h\|_\infty$$

which implies $\|M_h\| \leq \|h\|_\infty$, so M_h is a bounded linear functional. Therefore,

$$\begin{aligned} \langle M_h f, g\rangle &= \int_X fh\overline{g}d\mu \\ &= \int_X f\overline{\overline{h}g}d\mu \\ &= \langle f, \overline{h}g\rangle \\ &= \langle f, M_{\overline{h}}g\rangle \end{aligned}$$

So $M_h^* = M_{\overline{h}}$.

Theorem 3.37

Suppose V, W are Hilbert Spaces and let $T \in \mathscr{B}(V, W)$. Then the following are true:

1. $T^* \in \mathscr{B}(W, V)$
2. $(T^*)^* = T$
3. $\|T^*\|_{\mathscr{B}(W,V)} = \|T\|_{\mathscr{B}(V,W)}$

Definition 3.33

Let $T \in \mathscr{B}(V)$ where V is a Hilbert Space. Then T is **self adjoint** if $T = T^*$, i.e., $\forall f, g \in V$

$$\langle Tf, g\rangle = \langle f, Tg\rangle$$

Theorem 3.38

Let V be a Hilbert Space and $T \in \mathscr{B}(V)$. Assume $\langle Tf, f\rangle = 0,\ \forall f \in V$.

1. *If $\mathbb{F} = \mathbb{C}$*
2. *If $\mathbb{F} = \mathbb{R}$ and T is self-adjoint, $T = 0$.*

Theorem 3.39

Let $T \in \mathscr{B}(V)$, where V is a Hilbert Space over $\mathbb{C}$. Then T is self-adjoint if and only if $\langle Tf, f\rangle \in \mathbb{R},\ \forall f \in V$.

3.4.11 Operators

Definition 3.34

Let V be an NLS. A function $T : V \to V$ is called an **operator**.
If T is bounded, we write $T \in \mathscr{B}(V, V)$, or, more succinctly, $T \in \mathscr{B}(V)$.

Definition 3.35

An operator T is **invertible** if it is 1:1 and onto. We define the inverse as

$$T^{-1} : V \to V$$
$$\text{and}$$
$$T \circ T^{-1} = I : V \to V$$

Note: Since T is linear, T^{-1} is also linear.

Definition 3.36

Let $T \in \mathscr{B}(V)$ where V is a Hilbert Space.

1. T is **left invertible** iff $\exists S$ such that $ST = I$
2. T is **right invertible** iff $\exists S$ such that $TS = I$
3. if T is left and right invertible, T is invertible.
   (Suppose $S_1 T = I$ and $TS_2 = I$, then

$$S_1 T = I \Rightarrow S_1 T S_2 = S_2 \Rightarrow S_1 I = S_2 \Rightarrow S_1 = S_2)$$

Theorem 3.40

Let $T \in \mathscr{B}(V)$ where V is a Hilbert Space. T is left invertible iff $\exists \alpha \in (0, \infty)$ such that $\forall f \in V$,

$$\|f\| \le \alpha \|Tf\| \tag{3.1}$$

Theorem 3.41

Let $T \in \mathscr{B}(V)$ where V is a Hilbert Space. If T is left invertible, T^* is right invertible.

☞ **Remark**

Let $T \in \mathscr{B}(V)$ be invertible. Let V be a Banach Space. By the Open Mapping Theorem, T is an open map. Therefore T^{-1} is continuous, so $T^{-1} \in \mathscr{B}(V)$.

☞ **Remark**

By convention, we write:

1. $T : V \to V$
2. $T \circ T = TT = T^2 : V \to V$
3. $T \circ (T \circ T) = TTT = T^3 : V \to V$

Theorem 3.42

Let U, V, W be an NLS and $T \in \mathscr{B}(U, V)$, $S \in \mathscr{B}(V, W)$. Then

$$\|ST\| \le \|S\| \|T\|$$

Theorem 3.43

Let $T^k = T \circ T \circ T \circ \ldots \circ T$ (k times). Then

$$\|T^k\| \leq \|T\|^k$$

Theorem 3.44

Let $T \in \mathscr{B}(V)$ where V is a Banach Space. Assume $\|T\| < 1$. Then $I - T : V \to V$ is invertible and

$$(I-T)^{-1} = \sum_{k=0}^{\infty} T^k$$

Note: this is similar to the fact that for $z \in \mathbb{C}$ with $|z| < 1$,

$$\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k$$

Theorem 3.45

Let V be an NLS. Then V is a Banach Space if and only if, for every $\{g_k\}$ satisfying

$$\sum_{k=1}^{\infty} \|g_k\| < \infty$$

$\sum_{k=1}^{\infty} g_k$ *converges in V.*

Corollary 3.2

Suppose V is a Banach Space. The set of all invertible operators:

$$\mathscr{A} = \{T \in \mathscr{B}(V) : T \text{ is invertible}\}$$

is an open set in $\mathscr{B}(V)$.
Note: this implies the set of non-invertible operators in $\mathscr{B}(V)$ is closed, so a sequence of non-invertible operators converges.

3.4.12 Spectrum of an Operator

Definition 3.37

Let $T \in \mathscr{B}(V)$.

1. $\alpha \in \mathbb{F}$ is an **eigenvalue** of T if $T - \alpha I$ is not injective. (i.e. $(T - \alpha I) = 0,\ f \neq 0$ implies $Tf = \alpha f$.
2. $f \in V$ with $f \neq 0$ is an **eigenvector** of T corresponding to an eigenvalue of f, α if $Tf = \alpha f$
3. The **spectrum** of T is denoted $sp(T)$:

$$sp(T) = \{\alpha \in \mathbb{F} : T - \alpha I \text{ is not injective}\}$$

Remark on 1.

$T - \alpha I$ is injective if and only if $null(T - \alpha I) = \{0\}$. In other words, $T - \alpha I$ is not injective if and only if $\exists z \in V$ with $z \neq 0$ and $z \in null(T - \alpha I)$. Therefore, $(T - \alpha I)z = 0 \Rightarrow Tz = \alpha z$.

3.4.13 Compact Operator

Definition 3.38

An operator $T : V \to V$, where V is a Hilbert Space, is **compact** if for all bounded sequences $\{f_k\}_{k=1}^{\infty}$ in V, $\{Tf_k\}_{k=1}^{\infty}$ has a convergent subsequence.
We denote the set of compact operators on V as $C(V)$.

Theorem 3.46

Every compact operator on Hilbert Space is bounded, and therefore continuous.

3.4.14 Spectrum of A Compact Operator

Theorem 3.47

If $T : V \to V$ is compact on an infinite dimensional Hilbert Space V, then $0 \in sp(T)$.

Remark

The above implies that $T = T - 0I$ is not invertible, so T is not invertible.

Theorem 3.48

Let $T \in C(V)$ then $Range(T)$ cannot contain an infinite dimensional, closed subspace of V.

Example 3.10

Consider the measure space $([0,1]), \mathbb{B}, \lambda)$. and define $T : L^2([0,1]) \to L^2([0,1])$ by

$$Tf(x) = \int_0^1 K(x,y)f(y)dy$$

where $K \in C([0,1] \times [0,1])$ is a fixed kernel function. **We claim that T is a compact operator**.

Proof. First, note that

$$\begin{aligned} \|Tf\|_{L^2} &= \Big(\int_0^1 |Tf(x)|^2 dx\Big)^{1/2} \\ &\leq \|Tf\|_{L^\infty}\Big(\int_0^1 1dx\Big)^{1/2} \\ &= \|Tf\|_{L^\infty} \end{aligned}$$

Also note that, $\forall x \in [0,1]$,

$$\begin{aligned} |Tf(x)| &= \Big|\int_0^1 K(x,y)f(y)dy\Big| \\ &\leq K(x,y)\int_0^1 |f(y)|dy \end{aligned}$$

$$\leq K(x,y)\|f\|_{L^2}$$

So we have $\|Tf\|_{L^\infty([0,1])} \leq K(x,y)\|f\|_{L^2([0,1])}$, therefore

$$\|Tf\|_{L^2([0,1])} \leq \|Tf\|_{L^\infty([0,1])} \leq K(x,y)\|f\|_{L^2([0,1])}$$

So T is bounded, linear (by linearity of the integral), and maps $L^2([0,1])$ to $L^2([0,1])$, which means T is bounded operator.
Now let $\{f_n\}_{n=1}^{\infty}$ be a bounded sequence in $L^2([0,1])$. We want to show that $\{Tf_n\}$ has a convergent subsequence. In order to do this, we can show that Arzela-Ascoli applies:
Note that $\|f\|_2 \leq \|f\|_\infty$. Now, by the fact that $K \in C([0,1] \times [0,1])$, given $\epsilon > 0$, there exists $\delta > 0$ such that $\forall x, y, z \in [0,1]$, whenever $|x - z| < \delta$, $|K(x,y) - K(z,y)| < \epsilon$. So

$$\begin{aligned}|Tf_n(x) - Tf_n(z)| &\leq \int_0^1 |K(x,y) - K(z,y)||f_n(y)|dy \\ &< \epsilon \int_0^1 |f_n(y)|dy \\ &\leq \epsilon\|f_n\|_{L^2([0,1])} \\ &\leq C\epsilon\end{aligned}$$

which implies that $\{Tf_n\}$ is equicontinuous. We have already shown that $|Tf_n(x)| \leq \|K\|_{L^\infty([0,1]\times[0,1])} \int_0^1 |f_n(y)|dy \leq K(x,y)\|f_n\|_{L^2([0,1])}$, so $\{Tf_n\}$ is equibounded.
So, by Arzela-Ascoli, $\exists$ some subsequence of $\{Tf_n\}$ that converges uniformly to some g. But then

$$\begin{aligned}\|Tf_{n_k} - g\|_{L^2([0,1])} &= \Big(\int_0^1 |Tf_{n_k} - g|\Big)^{1/2} \\ &\leq \|Tf_{n_k} - g\|_{L^\infty([0,1])}\Big(\int_0^1 1dy\Big)^{1/2} \\ &= \|Tf_{n_k} - g\|_{L^\infty([0,1])} \to 0\end{aligned}$$

So T is compact. ■