Title:	Multiple-Source Multiple-Sink Maximum Flow in Directed Planar Graphs
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Multiple-Source Multiple-Sink Maximum Flow in Directed Planar Graphs

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Years aud Authors of Summarized Original Work

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Problem Definition

Given a directed, planar graph $G = (V, E)$ with arc capacities $c : E \to \Re^+$, a subset S of source vertices and a subset T of sink vertices, the goal is to find a maximum flow from the source vertices to the sink vertices:

$$\max \sum_{su: s\in S, su\in E} f_{su}$$

$$s.t. \sum_{uv: uv\in E} f_{uv} - \sum_{vw: vw\in E} f_{vw} = 0 \qquad \forall v \in V \setminus (S \cup T) \tag{1}$$

$$0 \le f_e \le c_e \qquad \forall e \in E \tag{2}$$

Key Results

In general (i.e., non-planar) graphs, multiple sources and sinks can be reduced to the single-source single-sink case by introducing an artificial source and sink and connecting them to all the sources and sinks, respectively, but this reduction does not preserve planarity. Using Orlin's algorithm for sparse graphs [21] leads to a running time of $O(n^2/\log n)$. For integer capacities less than U, one could instead use the algorithm of Goldberg and Rao [9], which leads to a running time of $O(n^{1.5}\log n \log U)$.

Maximum flow in planar graphs with multiple sources and sinks was first studied by Miller and Naor [19]. They gave a divide-and-conquer algorithm for the case where all the sinks and the sources are on the boundary of a single face. Plugging in the linear-time shortest-path algorithm of Henzinger et al. [12] yields a running time of $O(n \log n)$. Borradaile and Harutyunyan have given an iterative algorithm with the same running time [2]. Miller and Naor also gave an algorithm for the case where the sources and the sinks reside on the boundaries of k different faces. Using the $O(n \log n)$-time single-source single-sink maximum flow algorithm of Borradaile and Klein [3] yields a running time of $O(k^2 n \log^2 n)$. Miller and Naor show that, when it is known how much of the commodity is produced/consumed at each source and each sink, finding a consistent routing of flow that respects arc capacities can be reduced to negative-length shortest paths [19], which can be solved in planar graphs in $O(n \log^2 n/\log\log n)$ time [20].

Near-linear time algorithm

Borradaile et al. gave the first $O(n\, poly \log n)$-time algorithm for the multiple-source, multiple-sink maximum flow problem in directed planar graphs. The approach uses pseudoflows [10; 14] (flows which may violate the balance constraints (1) in a limited way) and a divide-and-conquer scheme influenced by that of Johnson and Venkatesan [15] and that of Miller and Naor [19], using the separators introduced by Miller: a (triangulated) planar graph G can be separated by a simple cycle C of $O(\sqrt{n})$ vertices [18].

In each of the two subgraphs, a more general problem is solved in which, after the two recursive calls have been executed, within each of the two subgraphs there is no residual path from any source to any sink nor from any source to C or from C to any sink. Then, since C is a separator, there is no residual path from any source to any sink in G, but however, the balance constraints (1) may not be satisfied for vertices in C. The flow is then balanced among the vertices in C by augmenting the flow so that there is no residual path in G from a vertex with positive inflow to a vertex with positive outflow. The resulting flow can then be turned into a maximum flow in linear time.

The core of the algorithm is this final balancing procedure which involves a series of $|C| - 1$ max-flow computations in G. Since $|C|$ is $O(\sqrt{n})$, the challenge is carrying out all these max-flow computations in near-linear time. The procedure uses a succinct representation to keep track of the changes to the pseudoflow without explicitly storing the changes. The representation relies on the relationship between circulations in G and shortest paths in the dual and the computations make use of an adaptation of Fakcharoenphol and Rao's efficient implementation of Dijkstra's algorithm [7]. The resulting running time to balance the flow is $O(n \log^2 n)$ time for an overall running time of $O(n \log^3 n)$-time for the original multiple-source, multiple-sink maximum flow problem.

Applications

Multiple-source multiple-sink min-cut arises in several computer-vision problems including image segmentation (or binary labeling) [11]. For the case of more than two labels, there is a powerful and effective heuristic [5] using very-large-neighborhood [1] local search; the inner loop consists of solving the two-label case.

The *maximum matching in a bipartite planar graph* reduces to multiple-source, multiple-sink maximum flow. Multiple-source, multiple-sink maximum flow can also be used for finding *orthogonal drawings of planar graphs with a minimum number of bends* [6] and *uniformly monotone subdivisions of polygons* [23].

Cross-References

Maximum *st*-flow in directed planar graphs
Maximum flow

Recommended Reading

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