Flow

1. For this question, assume you are given a directed graph G with capacities $c(e)$ on the edges and two identified vertices, a source s and a sink t.

   Let (S, T) and (S', T') be minimum st-cuts. Prove that $(S \cap S', T \cup T')$ and $(S \cup S', T \cap T')$ are also minimum st-cuts.

2. Describe an efficient algorithm to check whether a given flow network contains a unique maximum flow.

3. We define the *Escape Problem* as follows. We are given a directed graph $G = (V, E)$ (picture a network of roads). A certain collection of nodes $X \subset V$ are designated as *populated* nodes, and a certain other collection $S \subset V$ are designated as *safe nodes*. (Assume that X and S are disjoint.) In case of an emergency, we want evacuation routes from the populated nodes to the safe nodes. A set of evacuation routes is defined as a set of paths in G so that:

   - each node in X is the start of a path
   - each path ends in S
   - the paths do not share any edges

   Such a set of edges can be viewed as a way for the occupants of the populated nodes to escape to the safe nodes, without causing congestion on any of the edges.

   (a) Given G, X and S, show how to decide, in polynomial time whether such a set of evacuation routes exists.

   (b) Suppose we have exactly the same problem as in (a), but we strengthen *the paths do not share any edges* to *the paths do not share any nodes*. With this stronger condition, show how to decide whether such a set of evacuation paths exists.

   (c) Provide an example graph and sets S and X such that the answer to (a) is "yes" but the answer to (b) is "no".

4. Consider a set of mobile computing clients who need to be connected to one of several possible base stations. There are n clients, each specified by its (x, y) coordinates in the plane. There are also k base stations, also specified by their (x, y) coordinates in the plane.

   For each client, we wish to connect it to exactly one of the base stations. Our choice of connections is constrained in the following ways. There is a range parameter r: a client can only be connected to a base station that is within distance r. There is also a load parameter L: each base station can serve only L clients.

   Design (describe and analyze) a polynomial-time algorithm for the following problem: given the positions of the clients and a set of base stations, as well as the load and range parameters, decide whether every client can be connected simultaneously to a base station, subject to the described load and range conditions.

5. A cycle cover of a given directed graph $G = (V, E)$ is a set of vertex-disjoint cycles that cover all the vertices. Describe and analyze an efficient algorithm to find a cycle cover for a given graph, or correctly report that no cycle cover exists. *[Hint: Use bipartite matching!]*

6. Given an undirected graph $G = (V, E)$, with three vertices u, v, and w, describe and analyze an algorithm to determine whether there is a simple path from u to w that passes through v.