Linear Programming

1. *Hollywood.* A film producer is seeking actors and investors for his new movie. There are n available actors; actor i charges s_i dollars. For funding, there are m available investors. Investor j will provide p_j dollars, but only on the condition that certain actors $L_j \subseteq \{1, 2, \ldots, n\}$ are included in the cast (*all* of these actors L_j must be chosen in order to receive funding from investor j).

   The producer's profit is the sum of the payments from investors minus the payments to actors. The goal is to maximize this profit.

   (a) Express this problem as an integer linear program in which the variables take on values $\{0, 1\}$.

   (b) Now relax this to a linear program, and show that there must in fact be an *integral* optimal solution (as is the case, for example, with maximum flow and bipartite matching).

2. A matrix $A = (a_{ij})$ is skew-symmetric if and only if $a_{ji} = -a_{ij}$ for all indices $i \neq j$; in particular, every skew-symmetric matrix is square. A canonical linear program $\max\{c.x | Ax \leq b; x \geq 0\}$ is *self-dual* if the matrix A is skew-symmetric and the objective vector c is equal to the constraint vector b.

   (a) Prove any self-dual linear program Π is equivalent to its dual program $\amalg$.

   (b) Show that any linear program Π with d variables and n constraints can be transformed into a self-dual linear program with $n + d$ variables and $n + d$ constraints. The optimal solution to the self-dual program should include both the optimal solution for Π (in d of the variables) and the optimal solution for the dual program $\amalg$ (in the other n variables).

3. (a) Model the maximum-cardinality bipartite matching problem as a linear programming problem. The input is a bipartite graph $G = (U \cup V; E)$, where $E \subseteq U \times V$; the output is the largest matching in G. Your linear program should have one variable for each edge.

   (b) Now dualize the linear program from part (a). What do the dual variables represent? What does the objective function represent? What problem is this!?