CS 325 Visible Lines Notes
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Let $n \in \mathbb{N}$ and $m_1 < \cdots < m_n$. Now, for each $i \in \{1, \ldots, n\}$, let $b_i \in \mathbb{R}$ and $y_i(x) = m_i x + b_i$, for each $x \in \mathbb{R}$.

Definition. A point $(x, y_i(x))$ is a *visible point* on y_i if for each j, $y_i(x) \geq y_j(x)$. A line y_i is a *visible line* if there exists a visible point on y_i.

Note. Let $i \neq j$. Then the intersection of lines y_i and y_j is the unique point on y_i and y_j with x-coordinate $x_{i,j} = (b_i - b_j)/(m_j - m_i)$. Moreover, if $i < j$, then

$$x \leq x_{i,j} \Rightarrow y_i(x) \geq y_j(x) \tag{1}$$

$$x \geq x_{i,j} \Rightarrow y_i(x) \leq y_j(x) \tag{2}$$

Claim 1. y_1 and y_n are visible lines.

Proof. Let y_i be the first line that intersects y_1—as x increases. Then by (1), we have $y_1(x_{1,i}) \geq y_j(x_{1,i})$, for all j, and so y_1 is a visible line. Now, let y_i be the last line that intersects y_n—as x increases. Then by (2), we have $y_n(x_{i,n}) \geq y_j(x_{i,n})$, for all j, and so y_n is a visible line. ■

Note. Let y_i be a visible line and $(x, y_i(x))$ a visible point on y_i. Then

$$i < n \Rightarrow \exists\ j > i \text{ s.t. } x \leq x' \leq x_{i,j} \Rightarrow (x', y_i(x')) \text{ is a visible point on } y_i \tag{3}$$

$$1 < i \Rightarrow \exists\ j < i \text{ s.t. } x_{i,j} \leq x' \leq x \Rightarrow (x', y_i(x')) \text{ is a visible point on } y_i \tag{4}$$

Claim 2. If y_i is not visible, then there exist j, k with $j < i < k$ such that $y_j(x_{j,k}) > y_i(x_{j,k})$.

Proof. Suppose y_i is not visible. Then by claim 1, it follows that $1 < i < n$. Let j be the greatest index with $j < i$ such that y_j is visible and let k be the least index with $k > i$ such that y_k is visible. Since y_j is visible, there exists x such that $(x, y_j(x))$ is a visible point on y_j. Moreover, since $j < i < n$, it follows by (3) that there exists $\ell > j$ such that for each x' with $x \leq x' \leq x_{j,\ell}$, $(x', y_j(x'))$ is a visible point on y_j. Since y_i is not visible and k is the least index with $k > i$ such that y_k is a visible line, it must be the case that $\ell = k$ and so $(x_{j,k}, y_j(x_{j,k}))$ is a visible point on y_j. In particular, $y_j(x_{j,k}) > y_i(x_{j,k})$. ■

Claim 3. If there exist j, k with $j < i < k$ such that $y_j(x_{j,k}) > y_i(x_{j,k})$, then y_i is not visible.

Proof. Suppose there exist j, k with $j < i < k$ such that $y_j(x_{j,k}) > y_i(x_{j,k})$. First, we will show $x_{i,k} < x_{j,i}$. Since $i < k$ and $y_k(x_{j,k}) > y_i(x_{j,k})$, it follows by (1) that $x_{i,k} < x_{j,k}$. Since $j < i$ and $y_j(x_{j,k}) > y_i(x_{i,k})$, it follows by (2) that $x_{j,k} < x_{j,i}$ and so $x_{i,k} < x_{j,i}$. Now, we will show y_i is not visible. Choose an arbitrary x. Then either $x < x_{j,i}$ or $x \geq x_{j,i}$. If $x < x_{j,i}$, then by (1) we have $y_j(x) \geq y_i(x)$ and we're done. So, assume $x \geq x_{j,i}$. Since $x_{i,k} < x_{j,i}$, it follows that $x > x_{i,k}$ and so by (2) we have $y_k(x) \geq y_i(x)$. ■

Note. Suppose $j < k$. Then it follows by elementary algebra that $y_j(x_{j,k}) > y_i(x_{j,k})$ if and only if $m_j(b_j - b_k) + b_j(m_k - m_j) > m_i(b_j - b_k) + b_i(m_k - m_j)$. If all slopes and y-intercepts are integers, then this means we may test if $y_j(x_{j,k}) > y_i(x_{j,k})$ using only addition, subtraction, and multiplication of integers, i.e., no division.