Dijkstra’s algorithm: Correctness by induction

We prove that Dijkstra’s algorithm (given below for reference) is correct by induction. In the following, $G$ is the input graph, $s$ is the source vertex, $\ell(uv)$ is the length of an edge from $u$ to $v$, and $V$ is the set of vertices.

\begin{verbatim}
Dijkstra($G, s$)
for all $u \in V \setminus \{s\}$, $d(u) = \infty$
$d(s) = 0$
$R = \{\}$
while $R \neq V$
    pick $u \notin R$ with smallest $d(u)$
    $R = R \cup \{u\}$
    for all vertices $v$ adjacent to $u$
        if $d(v) > d(u) + \ell(u, v)$
            $d(v) = d(u) + \ell(u, v)$
\end{verbatim}

Let $d(v)$ be the label found by the algorithm and let $\delta(v)$ be the shortest path distance from $s$-to-$v$. We want to show that $d(v) = \delta(v)$ for every vertex $v$ at the end of the algorithm, showing that the algorithm correctly computes the distances. We prove this by induction on $|R|$ via the following lemma:

**Lemma:** For each $x \in R$, $d(x) = \delta(x)$.

**Proof by Induction:**

*Base case ($|R| = 1$):* Since $R$ only grows in size, the only time $|R| = 1$ is when $R = \{s\}$ and $d(s) = 0 = \delta(s)$, which is correct.

*Inductive hypothesis:* Let $u$ be the last vertex added to $R$. Let $R' = R \cup \{u\}$. Our I.H. is: for each $x \in R'$, $d(x) = \delta(x)$.

*Using the I.H.:* By the inductive hypothesis, for every vertex in $R'$ that isn’t $u$, we have the correct distance label. We need only show that $d(u) = \delta(u)$ to complete the proof.

Suppose for a contradiction that the shortest path from $s$-to-$u$ is $Q$ and has length $\ell(Q) < d(u)$. $Q$ starts in $R'$ and at some leaves $R'$ (to get to $u$ which is not in $R'$). Let $xy$ be the first edge along $Q$ that leaves $R'$. Let $Q_x$ be the $s$-to-$x$ subpath of $Q$. Clearly:

$$\ell(Q_x) + \ell(xy) \leq \ell(Q).$$

Since $d(x)$ is the length of the shortest $s$-to-$x$ path by the I.H., $d(x) \leq \ell(Q_x)$, giving us

$$d(x) + \ell(xy) \leq \ell(Q_x).$$

Since $y$ is adjacent to $x$, $d(y)$ must have been updated by the algorithm, so

$$d(y) \leq d(x) + \ell(xy).$$

Finally, since $u$ was picked by the algorithm, $u$ must have the smallest distance label:

$$d(u) \leq d(y).$$

Combining these inequalities in reverse order gives us the contradiction that $d(x) < d(u)$. Therefore, no such shorter path $Q$ must exist and so $d(u) = \delta(u)$.

This lemma shows the algorithm is correct by “applying” the lemma for $R = V$. \hfill $\blacksquare$