CS523: Advanced Algorithms **Spring 2014**

Lecture 2: MAXCUT for planar graphs

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2.1 Planar Graphs and Duality

Definition 2.1 *A graph $G = (V, E)$ is* planar *if it can be drawn on a plane in a way that it edges only intersect at their endpoints. Such a drawing is called a* planar embedding *of the graph G.*

Herein, when we talk about a planar graph G, we mean G and its planar embedding.

Definition 2.2 *Given a planar graph $G = (V, E)$, a* dual *graph of G, denoted by $G^* = (V^*, E^*)$, is a graph that each vertex corresponds to a face of G and an edge between two vertices corresponds to the edge between two neighboring faces.*

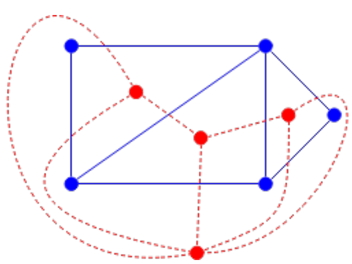

Figure 2.1: A planar graph (blue) and its dual (red)[1]

Lemma 2.3 $G = (G^*)^*$.

For a given edge e, we define:

- G/e is the graph obtained from *contracting* an edge e of G.
- $G - e$ is the graph obtained from *deleting* an edge e of G.

Lemma 2.4 *For a planar graph G and for any edge e of G that is not a loop, we have $(G/e)^* = G^* - e^*$ and $(G - e)^* = G^*/e^*$.*

[1]Source: `http://en.wikipedia.org/wiki/File:Duals_graphs.svg`

For a subset of vertices S of G, we define $\delta_G(S)$ to be a set of edges with exactly one endpoint in S. A set of edges of G is a *cut* if it has the form $\delta_G(S)$. A cut $\delta_G(S)$ is a *bond* if both $G[S]$ and $G[V\backslash S]$ are connected.

Lemma 2.5 *For a planar graph G, a subgraph C is a cycle of G if and only if C^* is a bond of G^*.*

2.2 Finding MAXCUT

We gives a reduction from the MAXCUT problem of a planar graph G to the maximum matching problem.

Definition 2.6 *An edge set D is an* odd-circuit cover *if its removal leaves a subgraph free of odd circuit.*

Observation 1 *If D is an odd-circuit cover, then every edge set D' such that $D \subseteq D'$ is also an odd-circuit cover.*

For an edge set $D \subseteq E$ of G, its complement is denoted by $\overline{D} = E\backslash D$

Observation 2 $w(D) + w(\overline{D}) = w(E)$

Lemma 2.7 *An edge set is contained in a cut if and only if its complement is an odd-circuit cover.*

Proof:

($\Rightarrow$) Let $\delta_G(S)$ be a cut of G, then the graph $G' = G(V, \delta_G(S))$ is a bipartite graph. Since a bipartite graph contains *no* odd cycle, $E\backslash\delta_G(S)$ is an odd-circuit cover. Therefore, for any subset $D \subseteq \delta_G(S)$, by Observation 1, $\overline{D}$ is an odd-circuit cover.

($\Leftarrow$) Let $D \subseteq E$ be an edge set of G. Since $\overline{D}$ is an odd-circuit cover, by definition of odd-circuit cover, $D = E\backslash\overline{D}$ contains no odd cycle. Therefore, the subgraph $G' = G(V, D)$ induced by D is a bipartite graph. Hence, D is contained in a cut. ■

Combining Observation 2 with Lemma 2.7, we get

Corollary 2.8 *An edge set is a maximum cut if and only if its complement is a minimum odd-circuit cover.*

A vertex v in G is called an *odd vertex* if it has odd degree.

Definition 2.9 *An edge set P is an* odd-vertex pairing *if its contraction leaves a multigraph free of odd vertices.*

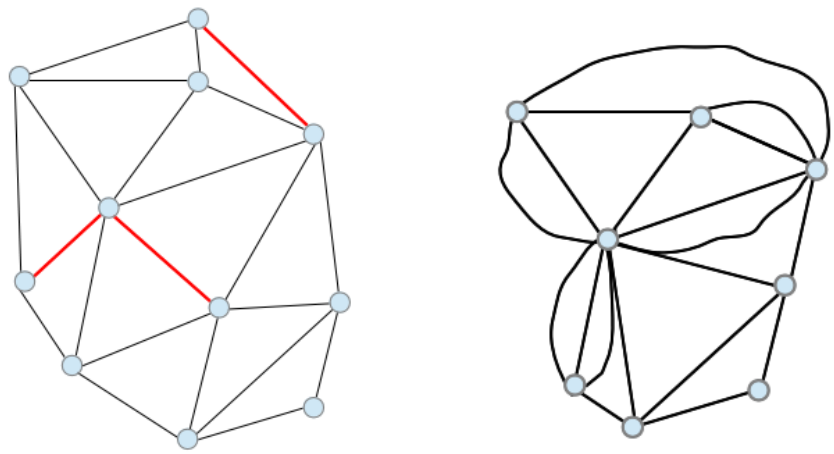

Figure 2.2: Odd-vertex pairing and its contraction.

Since the graph free of odd vertices is Eulerian, contracting an odd-vertice pairing results in an Eulerian multigraph

Lemma 2.10 *An edge set D is an odd-circuit cover of a planar graph G if and only if D^* is an odd-vertex pairing of G^**

Proof: We give a proof for the forward direction. The backward direction of the lemma is proved similarly. Let D be an odd-circuit cover, then $G-D$ contains no odd cycle. Therefore, by Lemma 2.5, G^*/D^* contains no odd cut. In other words, every vertex in G^*/D^* has even degree. Hence, D^* is an odd-vertex pairing of G^*. ∎

Lemma 2.11 *For an edge set P of an arbitrary multigraph G, P is a minimum odd-vertex pairing if and only if P is the collection of edge-disjoint paths with odd vertices in G as endpoints, using each once as endpoint, with minimum sum of path lengths.*

Proof: We prove that P can be decomposed into a collection of edge-disjoint paths, each path contains exactly two odd vertices of G which are endpoints of that path. Since P is an odd-vertex pairing, by definition, vertices in $G' = G(V, E/P)$ have even degree. Therefore, if a vertex v in G has odd degree, in P it also has odd degree. We decompose P by repeatedly applying the following *decomposition process*: for a connected component C_P of P, pick a pair of odd vertices (u, v) in C_P such that the shortest path between u and v in C_P contains no other odd vertex. Contracting that path leaves P_1 and G_1 in which P_1 is an odd-vertex pairing of G_1. Applying the decomposition process to P_1 and G_1 until there is no odd vertex left, we get a decomposition of P into collection of paths between odd vertices.

We prove that the collection of paths are edge-disjoint by contradiction. Assume that there are two pairs of odd vertices (u_1, v_1) and (u_2, v_2) such that the paths between (u_1, v_1) and (u_2, v_2) in $\mathcal{C}_P$ are not edge-disjoint (Figure 2.3). Then there are two new paths (u_1, u_2) and (v_1, v_2) such that they are disjoint and:

$$\ell_P(u_1, u_2) + \ell_P(v_1, v_2) < \ell_P(u_1, v_1) + \ell_P(u_2, v_2)$$

that contradicts to the minimality of P.

■

By Corollary 2.8 and Lemma 2.10, we reduce MAXCUT problem of G to finding minimum odd-vertex pairing of the dual graph G^* and by Lemma 2.11, we reduce to finding a minimum collection of edge-disjoint paths between odd vertices of the dual graph. Now we further reduce to the maximum matching in a complete graph.

Given a multigraph G, let G_c be a complete graph with vertex set is the set of odd vertices of G. For any pair of vertices u, v of G_c, let $e_c(u, v) = W - d_G(u, v)$ be the weight of edge (u, v) in G_c where $W = 1 + max\{d_G(u, v)|u, v \text{ are odd vertices of } G\}$. Since the number of vertices in G_c is even, the maximum matching is a perfect matching of G_c

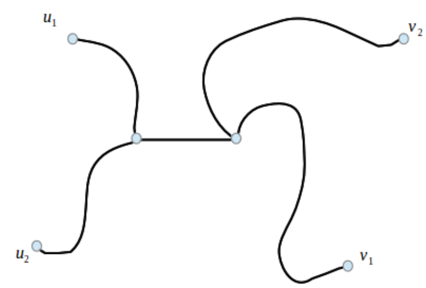

Figure 2.3: Two paths between (u_1, v_1) and (u_2, v_2)

Lemma 2.12 *The weight of maximum perfect matching of G_c is the weight of the minimum collection of edge-disjoint paths between odd vertices in G.*

Proof: Clearly, the weight of maximum perfect matching of G_c is the weight of the minimum collection P of shortest paths between odd vertices in G. The proof for the edge-disjointedness property is exactly the same to the proof of Theorem 2.11 ■

Maximum matching can be solved in polynomial time, therefore, MAXCUT in planar graphs can be solved in polynomial time.

References

[1] F. Hadlock. Finding a maximum cut of a planar graph in polynomial time. *SIAM J. Comput.*, 4(3):221–225, 1975.