CS325: Revisit Visibility Project – Solutions

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Proof of Claim 3

To prove:

Claim 3: If $\{z_1, z_2, \ldots, z_t\}$ and $\{z'_1, z'_2, \ldots, z'_s\}$ are two visible set of lines (each ordered by increasing slope), then the visible subset of $\{z_1, z_2, \ldots, z_t\} \cup \{z'_1, z'_2, \ldots, z'_s\}$ is $\{z_1, \ldots, z_i\} \cup \{z'_j, \ldots, z'_s\}$ for some $i \geq 1$ and $j \leq s$.

We will invoke, from Project 1:

Claim 1: y_i is not visible if and only if then there exist j, k with $j < i < k$ such that $y_j(x_{j,k}) > y_i(x_{j,k})$ where $(x_{j,k}, y_j(x_{j,k}))$ is the point of intersection of the lines y_j and y_k.

Claim 2: If $\{y_{j_1}, y_{j_2}, \ldots, y_{j_t}\}$ is the visible subset of $\{y_1, y_2, \ldots, y_{i-1}\}$ $(t \leq i-1)$ then $\{y_{j_1}, y_{j_2}, \ldots, y_{j_k}, y_i\}$ is the visible subset of $\{y_1, y_2, \ldots, y_i\}$ where y_{j_k} is the last line such that $y_{j_k}(x^*) > y_i(x^*)$ where $(x^*, y_{j_k}(x^*))$ is the point of intersection of the lines y_{j_k} and $y_{j_{k-1}}$.

Proof of Claim 3: For any z'_k in the second set of lines, the visible subset of $\{z_1, z_2, \ldots, z_t\} \cup \{z'_k\}$ is $\{z_1, z_2, \ldots, z_{i(k)}, z'_k\}$ by Claim 2 where $i(k)$ is the index of the last visible line of the first set after adding z'_k; $i(k)$ is a function of the index k. Let i^* be the minimum index $i(k)$ taken over all k (formally, $i^* = \min_{k=1,\ldots,s} i(k)$); put another way z_{i^*} is the last line of the first set such that $\{z_1, z_2, \ldots, z_{i^*}\}$ are part of the visible subset of $\{z_1, z_2, \ldots, z_t\} \cup \{z'_k\}$ for all $k = 1, \ldots, s$. Put yet another way, Claim 2 and our choice of i^* guarantees that $\{z_{i^*+1}, z_{i^*+2}, \ldots, z_t\}$ are *not* visible in the union of the first and second sets; note that this set is empty if $i^* = t$.

We similarly pick out an index j^* for the second set. Notice that if we take a mirror image of both sets of lines, the second set can *act as* the first set and the first set can *act as* the second set. Put another way, Claim 2 in its symmetric form is also true; that is, there is an index $j(\ell)$ such that $\{z_\ell, z'_{j(\ell)}, z'_{j(\ell)+1}, \ldots, z'_s\}$ is the visible subset of $\{z_\ell\} \cup \{z'_1, z'_2, \ldots, z'_s\}$. As above, we define $j^* = \min_{\ell=1,\ldots,t} j(\ell)$. This guarantees that $\{z'_{j^*}, z'_{j^*+1}, \ldots, z'_s\}$ are all visible in $\{z_\ell\} \cup \{z'_1, z'_2, \ldots, z'_s\}$ for every ℓ and that $\{z'_1, z'_2, \ldots, z'_{j^*-1}\}$ are not visible in the union of the first and second sets; note that this not visible set is be empty if $j^* = 1$.

So, we have argued that i^* and j^* are indices such that $\{z_{i^*+1}, z_{i^*+2}, \ldots, z_t\}$ and $\{z'_{j^*}, z'_{j^*+1}, \ldots, z'_s\}$ are *not* visible in $\{z_1, z_2, \ldots, z_t\} \cup \{z'_1, z'_2, \ldots, z'_s\}$. We complete the proof by arguing that $\{z_1, \ldots, z_{i^*}\} \cup \{z'_{j^*}, \ldots, z'_s\}$ is a visible set. For a contradiction, suppose z_i is not visible in the final set for some $i \leq i^*$; by Claim 1, there would have to be a line of lesser slope (and therefore in $\{z_1, z_2, \ldots, z_{i-1}\}$ and a line of greater slope (and therefore in $\{z_{i+1}, \ldots, z_t, z'_1, z'_2, \ldots, z'_s\}$) than z_i that witnesses z_i's non-visibility. Since z_i is visible in the first set, the line of greater slope that witnesses z_i's non-visibility must be in the second set. Therefore z_i would be determined to be non-visible in $\{z_1, z_2, \ldots, z_t\} \cup \{z'_k\}$ for some k, contradicting our definition of i^*. The same argument, but symmetric, can be used to argue that z'_j is visible in the final set for all $j = j^*, j^*+1, \ldots, s$. $\square$

(Note that this proof contains extra explanations; correctness can be maintained while removing some of these redundant lines and thus shortening the proof.)

Merging Visible Lines

In our pseudocode, we use the following notation and observations:

- $\bar{z}_i$ represents the visible segment of z_i in $\{z_1, z_2, \ldots, z_t\}$; $\bar{z}'_j$ represents the visible segment of z'_j in $\{z'_1, z'_2, \ldots, z'_s\}$. Note that $\bar{z}_i$ can be determined by simple comparisons of z_i with z_{i+1} and z_{i-1} (likewise for $\bar{z}'_j$). Using basic geometry, we can determine if $\bar{z}_i$ and $\bar{z}'_j$ intersect in constant time *without using division* (it is a little complicated, and left for you to work out if you haven't already).

- We use $x_{\max}(\bar{z}_i)$ to represent the maximum x value for which z_i is visible in $\{z_1, z_2, \ldots, z_t\}$ and $x_{\max}(\bar{z}'_j)$ to represent the maximum x value for which z'_j is visible in $\{z'_1, z'_2, \ldots, z'_s\}$. Using basic geometry, we can compare $x_{\max}(\bar{z}_i)$ to $x_{\max}(\bar{z}'_j)$ in constant time *without using division.*

Our merging procedure relies on the fact that $\bar{z}_{i^*}$ and $\bar{z}'_{j^*}$ (as defined in the proof above) intersect. This also guarantees that our pseudocode will indeed reach the return statment.

```
MergeVisible({z_1, z_2, ..., z_t},{z'_1, z'_2, ..., z'_s})
    i ← 1, j ← 1
    while
        if z̄_i intersects z̄'_j
            return {z_1, z_2, ..., z_i} ∪ {z'_j, z'_2, ..., z'_s}
        if x_max(z̄_i) > x_max(z̄'_j)
            j ← j+1
        else (in this case x_max(z_i) ≤ x_max(z̄'_j))
            i ← i+1
```

The running time is $O(t+s)$ time because there are *at most* $t+s$ iterations of the while loop as at least one of the two indices is increased in each iteration and as we mentioned above, the comparisons can be implemented in $O(1)$ time. Note that the code correctly finds the visible set because every visible segment of the first set and every visible segment of the second set whose x-domains overlap are guaranteed to be compared (draw the x-domains of each set to see how the indices are increased correctly).

Divide-and-Conquer Visible Lines

```
Visible({y_1, y_2, ..., y_n})
    if n ≤ 2
        return {y_1, ..., y_n}
    ℓ ← ⌊n/2⌋
    return MergeVisible(Visible({y_1, y_2, ..., y_ℓ}),Visible({y_{ℓ+1}, y_{ℓ+2}, ..., y_n}))
```

The running time of `Visible` for a set of n lines is described by the recurrence relation

$$T(n) = 2T(n/2) + O(n).$$

Note that since the running time of `MergeVisible` is $O(n)$ (not $\Theta(n)$),

$$T(n) \leq 2T(n/2) + cn \text{ for some constant } c.$$

The solution to this is $T(n) = O(n \log n)$. Note that to really get O not Θ for the running time, the algorithm would need to be implemented so that the subsets of the arrays are passed and returned in terms of starting and ending points of visible sections. This is possible, but I certainly couldn't be bothered with the extra complication.

Claim: `Visible` correctly determines the visible subset of an input set of n lines.

Proof (by induction): For the base case of $n = 1$ or $n = 2$, all the lines are visible and `Visible` is correct.

For the inductive hypothesis, we assume that `Visible` correctly determines the visible subset of an input set of ℓ lines for any $\ell < n$.

There are two recursive calls to `Visible` when $n \geq 3$.

The first is on a set of size $\ell = \lfloor \frac{n}{2} \rfloor$ which is $< n$; therefore `Visible(`$\{y_1, y_2, \ldots, y_\ell\}$`)` returns the visible subset of $\{y_1, y_2, \ldots, y_\ell\}$ by the inductive hypothesis.

The second is on a set of size $n - \ell = n - \lfloor \frac{n}{2} \rfloor$ which is $< n$ since $n \neq 1$ when the recursive call is made; therefore `Visible(`$\{y_{\ell+1}, y_{\ell+2}, \ldots, y_n\}$`)` returns the visible subset of $\{y_{\ell+1}, y_{\ell+2}, \ldots, y_n\}$ by the inductive hypothesis.

Since $\{y_1, y_2, \ldots, y_n\}$ is sorted by increasing slope, the recursive calls return two sets satisfying the input requirements for `MergeVisible`. Since `MergeVisible` is correct, `Visible` correctly determins the visible subset of $\{y_1, y_2, \ldots, y_n\}$. $\square$