A Convergent Iteration Method for 3-D AOA Localization

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***Abstract*—Using angle-of-arrival (AOA) measurements from several sensors to locate a target in 3-D space is one of the commonly used wireless localization techniques. A challenge of this technique is that the weighted least-squares (WLS) needed for it tend to incur the threshold effect when the noise level is high or when the localization geometry is poor, because its implementation requires the inversion of a matrix whose condition number will be large under such conditions. This paper presents an iterative method that does not require any matrix operation and is proved to be convergent with any initial value. Additionally, it outperforms the WLS-based methods as validated by comprehensive simulation results.**

***Index Term*-Angle-of-arrival (AOA), target localization, weighted least-squares (WLS).**

I. Introduction

TARGET localization has found a large number of applications [1]. Many techniques have been used for target localization, such as time of arrival (TOA) [2], [3], time-difference of arrival (TDOA) [4], [5], received signal strength (RSS) [6], [7], angle of arrival (AOA) [8]–[10], and their hybrid [11]–[14]. TOA, TDOA, and RSS measurements generally provide the distance information between the target and sensors, while AOA measurements show the target orientations from the sensors [6]. TOA and TDOA based localization systems require accurate time synchronization, which increases the system complexity [15]. RSS-based systems assume that the received power follows an exponential-decay model and are commonly used for scenarios of short distances such as indoor localization [1]. AOA-based localization systems do not require time synchronization and have broader application scenarios.

2-D AOA localization has been well studied [16], [17]. However, unlike TOA, TDOA, and RSS, AOA-based 2-D localization algorithms cannot be straightforwardly extended to the 3-D space, and 3-D AOA localization is a much more difficult problem, and remains to be an open research problem.

This paper addresses target localization with 3-D AOA measurements. A main challenge for 3-D AOA localization is that the azimuth and elevation angle measurements are highly nonlinearly related to the target position, making it difficult to find the solution. Grid search in 3-D space is too time-consuming for practical applications. The Taylor series expansion (TSE) method could diverge when the initial value is not sufficiently close to the true location. Closed-form solutions have the advantage of a low computational complexity and a good estimation performance when the noise variance is sufficiently small.

In [8], Badriasl *et al.* firstly propose an improved pseudolinear estimator (IPLE) for the 3-D target motion analysis (TMA) using azimuth and elevation angles. However, the IPLE algorithm is not an unbiased estimator. In order to obtain an asymptotically unbiased estimator, they also develop an improved weighted instrumental variable estimator (IWIV). Wang *et al.* [9] firstly derive the pseudolinear equations, and then apply the weighted least-squares (WLS) to solve them. However, it is shown that the solution from WLS has a large bias due to the presence of measurement noise in both sides of the pseudolinear equations. In order to address the bias problem, the authors develop a bias reduced solution based on the WLS formulation (WLS-BR). The WLS and WLS-BR solutions tend to incur the threshold effect when the noise level is high or when the localization geometry is poor, because its implementation requires the inversion of a matrix whose condition number will be large under such conditions. In [10], Sun *et al.* propose an eigenspace solution for source localization in the modified polar representation (MPR). Sensor position errors are considered in this solution.

This paper presents a new iterative method that does not require any matrix operation and does not suffer from the threshold effect. The proposed method has following merits:

- It converges with any initial values.
- Its estimation performance is superior to those of the existing WLS and WLS-BR solutions when the noise level is high or when the localization geometry is poor.
- It does not need prior knowledge of the noise variance that WLS and WLS-BR solutions require.

We derive the computational complexity of the proposed algorithm, and compare its performance with two existing WLS-based schemes as well as the Cramér-Rao lower bound (CRLB).

The rest of this paper is organized as follows. Section II describes the measurement model of 3-D AOA localization. In Section III, we derive the proposed iterative method. Section IV proves the convergence of the proposed method. Section V analyzes the computational complexity of the proposed method. Section VI presents performance simulation results. Section VII concludes this paper.

II. AOA Measurements Model

Consider a sensor network of M sensor nodes with $\mathbf{s}_i = [x_i, y_i, z_i]^T$ being the known coordinates of the ith node. Let $\mathbf{u} = [x, y, z]^T$ be the unknown coordinates of the target. The

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azimuth and elevation angle measurements in a 3-D space are expressed as

$$\theta_i = \theta_i^0 + e_i, i = 1, 2, \ldots, M, \tag{1a}$$

$$\phi_i = \phi_i^0 + n_i, i = 1, 2, \ldots, M, \tag{1b}$$

where θ_i^0 and ϕ_i^0 are the unknown true values of the azimuth and elevation angles, respectively, and e_i and n_i are measurement errors, modeled as independent additive white Gaussian noise (AWGN) with variance σ^2. The true azimuth angle θ_i^0 can be expressed as

$$\theta_i^0 = \cos^{-1}\left(\frac{x - x_i}{\sqrt{(x - x_i)^2 + (y - y_i)^2}}\right), \tag{2a}$$

$$\theta_i^0 = \sin^{-1}\left(\frac{y - y_i}{\sqrt{(x - x_i)^2 + (y - y_i)^2}}\right), \tag{2b}$$

and the true elevation angle ϕ_i^0 is written as

$$\phi_i^0 = \cos^{-1}\left(\frac{\sqrt{(x - x_i)^2 + (y - y_i)^2}}{\sqrt{(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2}}\right), \tag{3a}$$

$$\phi_i^0 = \sin^{-1}\left(\frac{z - z_i}{\sqrt{(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2}}\right). \tag{3b}$$

For brevity, $\sqrt{(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2}$ will be replaced by $\|\mathbf{u} - \mathbf{s}_i\|$ in the following derivations.

The goal of the AOA localization problem is to use θ_i, ϕ_i and $\mathbf{s}_i$ to estimate the target position $\mathbf{u}$. Next we develop a simple and efficient iterative method to solve this problem.

III. Iterative Localization Algorithm

Taking cosine of both sides of (2a) and (3a), and taking sine of both sides of (2b) and (3b), we have

$$x = x_i + \|\mathbf{u} - \mathbf{s}_i\| \cos\phi_i^0 \cos\theta_i^0, \tag{4a}$$

$$y = y_i + \|\mathbf{u} - \mathbf{s}_i\| \cos\phi_i^0 \sin\theta_i^0, \tag{4b}$$

$$z = z_i + \|\mathbf{u} - \mathbf{s}_i\| \sin\phi_i^0, \tag{4c}$$

where $i = 1, \cdots, M$. This means that the unknown target position $\mathbf{u} = [x, y, z]^T$ must satisfy the above $3M$ equations simultaneously.

Now let us assume that an initial value of the target position $\mathbf{u}_{k-1} = [x_{k-1}, y_{k-1}, z_{k-1}]^T$ is available, and use the measurement angles θ_i and ϕ_i to replace the true angles. For the kth iteration of the ith sensor, we obtain the estimated target position as

$$x_{k,i} \approx x_i + \|\mathbf{u}_{k-1} - \mathbf{s}_i\| \cos\phi_i \cos\theta_i, \tag{5a}$$

$$y_{k,i} \approx y_i + \|\mathbf{u}_{k-1} - \mathbf{s}_i\| \cos\phi_i \sin\theta_i, \tag{5b}$$

$$z_{k,i} \approx z_i + \|\mathbf{u}_{k-1} - \mathbf{s}_i\| \sin\phi_i. \tag{5c}$$

Let the updated target position $\mathbf{u}_k = [x_k, y_k, z_k]^T$ be the centroid of $\mathbf{u}_{k,i} = [x_{k,i}, y_{k,i}, z_{k,i}]^T, i = 1, 2, \ldots, M$, i.e.,

$$\mathbf{u}_k = \frac{1}{M}\sum_{i=1}^{M} \mathbf{u}_{k,i} = \frac{1}{M}\sum_{i=1}^{M} [\mathbf{s}_i + \|\mathbf{u}_{k-1} - \mathbf{s}_i\| \, \mathbf{p}_i], \tag{6}$$

where $\mathbf{p}_i = [\cos\phi_i \cos\theta_i, \cos\phi_i \sin\theta_i, \sin\phi_i]^T$. The resulting $\mathbf{u}_k$ is applied as the input in the next iteration as the initial value. The kth updating process is sketched in Fig. 1, which shows that $\mathbf{u}_k$ is closer to $\mathbf{u}$ than $\mathbf{u}_{k-1}$ is.

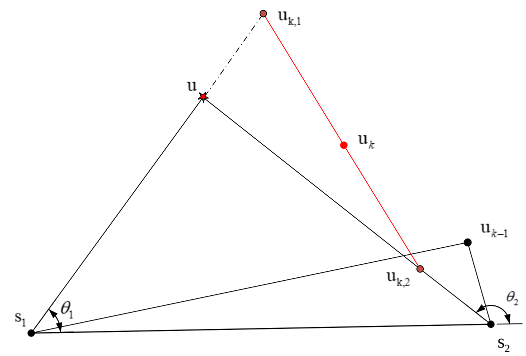

Fig. 1: Geometric illustration of the iteration process for the proposed algorithm with two sensor nodes in 2-D plane and $e_i = 0, i = 1, 2$ ($\mathbf{u}_{k,1}$ and $\mathbf{u}_{k,2}$ are intermediate variables).

A limit on the number of iterations, K, should be applied for practical scenarios. One proper criterion to stop the iteration is when the following condition is met:

$$\|\mathbf{u}_k - \mathbf{u}_{k-1}\| \le \epsilon, k = 1, 2, \cdots, K. \tag{7}$$

where ϵ is a predetermined threshold.

Without loss of generality, let the centroid of $\mathbf{s}_i$ be the initial value, i.e., $\mathbf{u}_0 = \frac{1}{M}\sum_{i=1}^{M} \mathbf{s}_i$. The pseudocode of the proposed iterative algorithm is shown in Algorithm 1.

Algorithm 1: Iterative Algorithm for 3-D AOA Localization

```
Data: θ_i, φ_i, s_i, K, ε;
Result: u
u_0 ← (1/M) Σ_{i=1}^{M} s_i;
u ← u_0;
k ← 1;
while k ≤ K do
    u_k ← calculate(6);
    if ||u_k − u|| > ε then
        u ← u_k;
    else
        break;
    k ← k + 1;
```

IV. Proof of Convergence

Proposition: The proposed iterative algorithm is guaranteed to converge with any initial values if $\mathrm{rank}([\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_M]) > 1$, or $\mathbf{s}_i$, $\mathbf{u}_k$ and $\mathbf{u}_{k-1}$ are not collinear.

Proof. Eq. (6) can be written as

$$\mathbf{u}_k = f(\mathbf{u}_{k-1}) = \frac{1}{M}\sum_{i=1}^{M} [\mathbf{s}_i + \|\mathbf{u}_{k-1} - \mathbf{s}_i\| \, \mathbf{p}_i], \tag{8}$$

and

$$\begin{aligned}
&\|f(\mathbf{u}_k) - f(\mathbf{u}_{k-1})\| \\
&= \left\| \frac{1}{M} \sum_{i=1}^{M} (\|\mathbf{u}_k - \mathbf{s}_i\| - \|\mathbf{u}_{k-1} - \mathbf{s}_i\|)\mathbf{p}_i \right\| \\
&\leq \frac{1}{M} \sum_{i=1}^{M} \|(\|\mathbf{u}_k - \mathbf{s}_i\| - \|\mathbf{u}_{k-1} - \mathbf{s}_i\|)\mathbf{p}_i\| \\
&= \frac{1}{M} \sum_{i=1}^{M} |(\|\mathbf{u}_k - \mathbf{s}_i\| - \|\mathbf{u}_{k-1} - \mathbf{s}_i\|)| \\
&\leq \frac{1}{M} \sum_{i=1}^{M} \|\mathbf{u}_k - \mathbf{u}_{k-1}\| = \|\mathbf{u}_k - \mathbf{u}_{k-1}\|,
\end{aligned} \tag{9}$$

where the two inequalities are deduced from triangle inequality, and the second equality uses the fact that $\|\mathbf{p}_i\| = 1$.

In the first inequality, the equal sign holds if and only if all the vectors $(\|\mathbf{u}_k - \mathbf{s}_i\| - \|\mathbf{u}_{k-1} - \mathbf{s}_i\|)\mathbf{p}_i$ are in the same direction. The equal sign in the second inequality holds if and only if $\mathbf{s}_i$, $\mathbf{u}_k$, and $\mathbf{u}_{k-1}$ are collinear.

Consequently, if only $\mathrm{rank}([\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_M]) > 1$, or $\mathbf{s}_i$, $\mathbf{u}_k$ and $\mathbf{u}_{k-1}$ are not collinear, for $\mathbf{u}_k \neq \mathbf{u}_{k-1}$,

$$\|f(\mathbf{u}_k) - f(\mathbf{u}_{k-1})\| < \|\mathbf{u}_k - \mathbf{u}_{k-1}\|, \tag{10}$$

i.e., the function $f(\cdot)$ is a contraction mapping.

According to the Banach fixed-point theorem [18], the proposed algorithm converges to a unique fixed point with any initial values. □

Remark 1. *The true target position is the unique fixed point of function $f(\cdot)$ in the absence of measurement noise, i.e., $\mathbf{u} = f(\mathbf{u})$.*

Remark 2. *The two sufficient conditions are easy to satisfy in practice.*

Remark 3. *The proof is derived from Banach fixed-point theorem, and hence the proposed algorithm is named as fixed point iteration (FPI).*

V. Complexity Analysis

The proposed iterative method is simple; each iteration requires only addition, multiplication, division, square-root, and some trigonometric operations. The number of various operations needed for each iteration are given in Appendix A. The computational complexities of the proposed method and two other methods are given in Table I. The average number of iterations is 532 for the proposed method (in the scenario of seven sensors). Although the proposed method has much higher running time than the WLS and WLS-BR solutions, its running time is still very low for implementation in practical scenarios (in milliseconds range in the scenario of seven sensors, CPU: 2.6 GHz, Intel Core i7).

VI. Simulation Results

In this section, we conduct several numerical simulations to assess the performance of the proposed AOA localization method, and compare it with those of the WLS and WLS-BR [9], and the CRLB. In the following simulations, except for Fig. 2, the initial value is set as $\mathbf{u}_0 = \frac{1}{M}\sum_{i=1}^{M} \mathbf{s}_i$. The maximum number of iterations and the threshold to stop the iteration for the proposed method are set to 1000 and 0.001 [1], respectively.

The target position is randomly placed in a $4000 \times 4000 \times 4000$ cube centered at $[0, 0, 0]^T$. A total of 100 random target positions are used to obtain the average CRLB, RMSE and the bias norm, and 1000 Monte Carlo realizations were generated for each target position. The azimuth and elevation angle errors are AWGN with variance σ^2.

First, seven sensors' positions are fixed and listed in Table II. Fig. 2 to Fig. 4 are drawn for this scenario.

TABLE I: Computational Complexity

Operation \ Algorithm	WLS	WLS-BR	Proposed
Multiplication	104M	143M	(6M+3)K+2M
division	2M	2M+3	3K
Trigonometric	4M	4M	4M
Square-Root	M	M	(M+1)K
3×3 Matrix Inversion	2	0	0
4×4 Generalized Eigen-decomposition	0	1	0

TABLE II: Sensors Position (m)

Sensors No. \ Positions	x	y	z
1	600	0	10
2	400	0	20
3	200	0	30
4	0	0	40
5	0	200	30
6	0	400	20
7	0	600	10

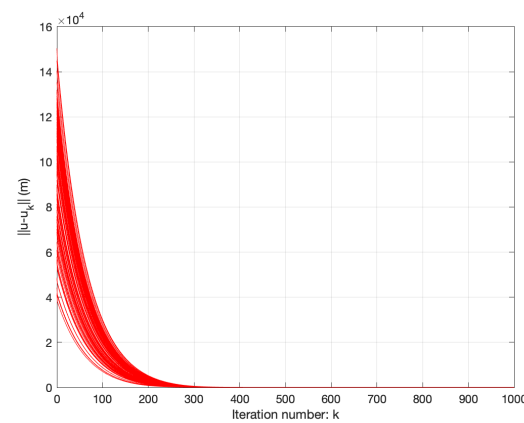

Fig. 2: Estimate bias vs. iteration number with 100 random initial values.

The following settings are applied in Fig. 2: $u = [500, 1000, 1500]^T m$, seven sensors, $\sigma = 1°$, 100 initial values are randomly placed in a $100000 \times 100000 \times 100000$ cube

[1]The stop threshold ϵ should be much smaller than 1. We have tested the influence of the parameter ϵ on the performance of the proposed algorithm in simulations. The results show that $\epsilon = 0.001, 0.01, 0.1$ have the same performance, and $\epsilon = 1$ results in a worse performance.

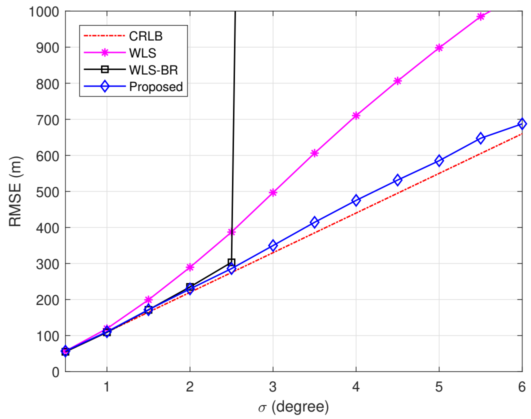

Fig. 3: RMSE vs. σ with seven sensors.

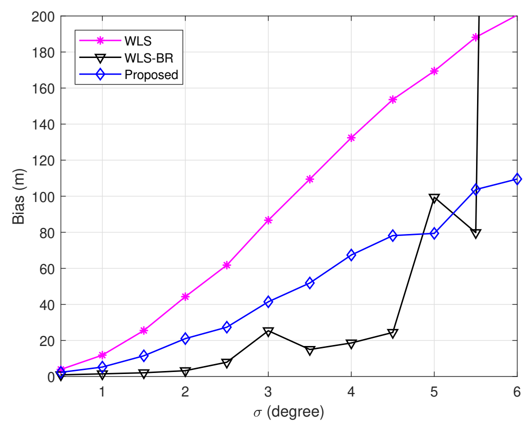

Fig. 4: Bias vs. σ with seven sensors.

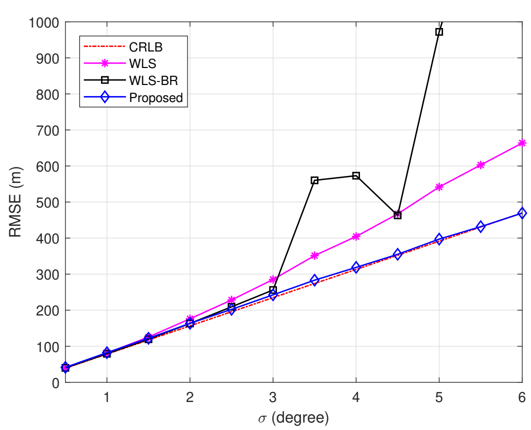

Fig. 5: RMSE vs. σ with six sensors.

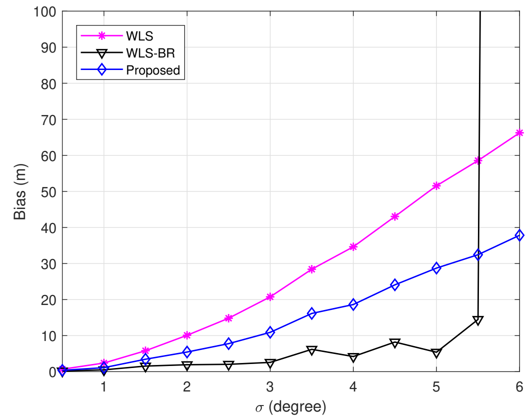

Fig. 6: Bias vs. σ with six sensors.

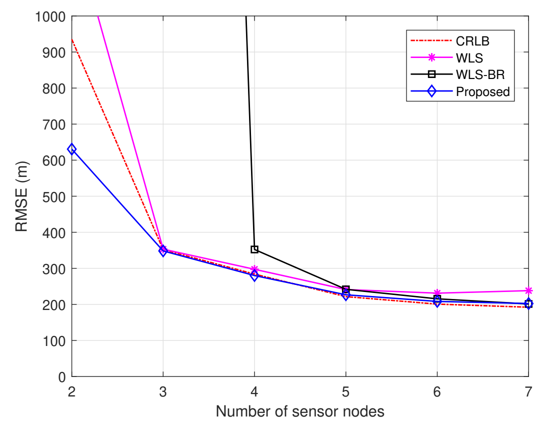

Fig. 7: RMSE vs. Number of sensor nodes with $\sigma = 3°$.

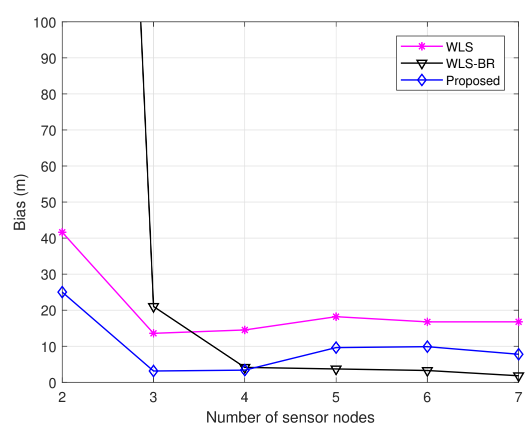

Fig. 8: Bias vs. Number of sensor nodes with $\sigma = 3°$.

centered at $[0,0,0]^T$. It shows that the proposed algorithm converges well regardless of the initial values.

Fig. 3 shows the RMSEs of the three algorithms with seven sensors versus the noise standard deviation σ. It is observed that the proposed method is superior to the other two algorithms and achieves a performance close to the CRLB. The WLS algorithm deviates from the CRLB when σ is larger than 1°, and the WLS-BR algorithm deviates from the CRLB when σ is larger than 2°. Additionally, the performance of the WLS-BR algorithm begins to deteriorate drastically when σ is larger than 2.5°. As pointed out in [9], the $v(4)$ is near zero when the noise level is too high or when the localization geometry is poor, leading to the threshold effect.

In Fig. 4, the biases of the three algorithms with seven sensors are provided. The results show that the WLS-BR algorithm has the lowest bias when σ is smaller than 4.5°, and the proposed algorithm has a lower bias than the WLS algorithm.

Second, the sensors' positions are randomly generated from $[-600,600]m\times[-600,600]m\times[-600,600]m$ in the 100 trials. Fig. 5 to Fig. 8 are drawn from this scenario.

Figs. 5 and 6 compare, respectively, the RMSEs and biases of the three algorithms for the scenario of six sensors. As expected, the RMSEs of the three estimators increase as the measurement error increases. The proposed algorithm is found to have the lowest RMSE, and the WLS-BR algorithm has the lowest bias when σ is smaller than 5.5°.

As a final comparison, Figs. 7 and 8 show, respectively, the RMSEs and biases of the three algorithms for the case that number of sensor nodes is varying. The WLS-BR has the poorest performance when number of sensor nodes is smaller than 3. This observation shows that a very small number of sensors (localization geometry is poor) leads to the threshold effect. Note that the RMSE of the proposed method is below the CRLB when number of sensor nodes is two. The plausible reason is that the CRLB is not applicable for biased estimators, and the bias of the proposed algorithm is great than 0 when there are only two sensor nodes.

VII. Conclusions

We have developed a convergent iterative method for 3D AOA localization in this paper. The proposed method has the advantage of simple operations (no matrix inversion or generalized eigenvalue decomposition is required) and it performs better than existing schemes, especially when the number of sensors is small or when the noise level is high.

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Appendix A

The computational complexity of each iteration of the proposed method is listed as follows:

- Eq. (5): $6M$ multiplications, $4M$ trigonometric and M square-root operations.
- Eq. (6): 3 divisions.
- Eq. (7): 3 multiplications and 1 square-root operation.

The computational complexity for the proposed method is $(6M+3)K+2M$ (2M is computed from $\cos\phi_i\cos\theta_i$ and $\cos\phi_i\sin\theta_i$ in (5)) multiplications, $3K$ divisions, $4M$ trigonometric and $(M+1)K$ square-root operations.

References

[1] F. Zafari, A. Gkelias, and K. K. Leung, "A survey of indoor localization systems and technologies," *IEEE Commun. Surv. Tut.*, vol. 21, no. 3, pp. 2568–2599, 2019.

[2] A. Beck, P. Stoica, and J. Li, "Exact and approximate solutions of source localization problems," *IEEE Trans. Signal Process.*, vol. 56, no. 5, pp. 1770–1778, 2008.

[3] Y. Zou and H. Liu, "A simple and efficient iterative method for TOA localization," in *Proc. IEEE Int. Conf. Acoust. Speech, Signal Process.*, 2020, pp. 4881–4884.

[4] Y.-T. Chan, H. Y. C. Hang, and P.-c. Ching, "Exact and approximate maximum likelihood localization algorithms," *IEEE Trans. Veh. Technol.*, vol. 55, no. 1, pp. 10–16, 2006.

[5] Z. Su, G. Shao, and H. Liu, "Semidefinite programming for nlos error mitigation in tdoa localization," *IEEE Commun. Lett.*, vol. 22, no. 7, 2018.

[6] H. C. So and L. Lin, "Linear least squares approach for accurate received signal strength based source localization," *IEEE Trans. Signal Process.*, vol. 59, no. 8, pp. 4035–4040, 2011.

[7] Y. Zou and H. Liu, "RSS-based target localization with unknown model parameters and sensor position errors," *IEEE Trans. Veh. Technol.*, vol. 70, no. 7, pp. 6969–6982, 2021.

[8] L. Badriasl and K. Dogancay, "Three-dimensional target motion analysis using azimuth/elevation angles," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 50, no. 4, pp. 3178–3194, 2014.

[9] Y. Wang and K. Ho, "An asymptotically efficient estimator in closed-form for 3-D AOA localization using a sensor network," *IEEE Trans. Wireless Commun.*, vol. 14, no. 12, pp. 6524–6535, 2015.

[10] Y. Sun, K. Ho, and Q. Wan, "Eigenspace solution for AOA localization in modified polar representation," *IEEE Trans. Signal Process.*, vol. 68, pp. 2256–2271, 2020.

[11] L. Taponecco, A. A. D'Amico, and U. Mengali, "Joint TOA and AOA estimation for UWB localization applications," *IEEE Trans. Wireless Commun.*, vol. 10, no. 7, pp. 2207–2217, 2011.

[12] J. Yin, Q. Wan, S. Yang, and K. Ho, "A simple and accurate TDOA-AOA localization method using two stations," *IEEE Signal Process. Lett.*, vol. 23, no. 1, pp. 144–148, 2016.

[13] T. Jia, H. Wang, X. Shen, Z. Jiang, and K. He, "Target localization based on structured total least squares with hybrid TDOA-AOA measurements," *Signal Processing*, vol. 143, pp. 211–221, 2018.

[14] S. Tomic, M. Beko, and R. Dinis, "3-D target localization in wireless sensor networks using RSS and AOA measurements," *IEEE Trans. Veh. Technol.*, vol. 66, no. 4, pp. 3197–3210, 2016.

[15] Y. Zou, H. Liu, and Q. Wan, "Joint synchronization and localization in wireless sensor networks using semidefinite programming," *IEEE Internet of Things Journal*, vol. 5, no. 1, pp. 199–205, 2017.

[16] Z. Wang, J.-A. Luo, and X.-P. Zhang, "A novel location-penalized maximum likelihood estimator for bearing-only target localization," *IEEE Trans. Signal Process.*, vol. 60, no. 12, pp. 6166–6181, 2012.

[17] Q. Zhou and Z. Duan, "Weighted intersections of bearing lines for aoa based localization," in *17th International conference on information fusion (FUSION)*. IEEE, 2014, pp. 1–8.

[18] M. A. Khamsi and W. A. Kirk, *An introduction to metric spaces and fixed point theory*. John Wiley & Sons, 2011, vol. 53.