

# Linear Algebra & Geometry

why is linear algebra useful in computer vision?

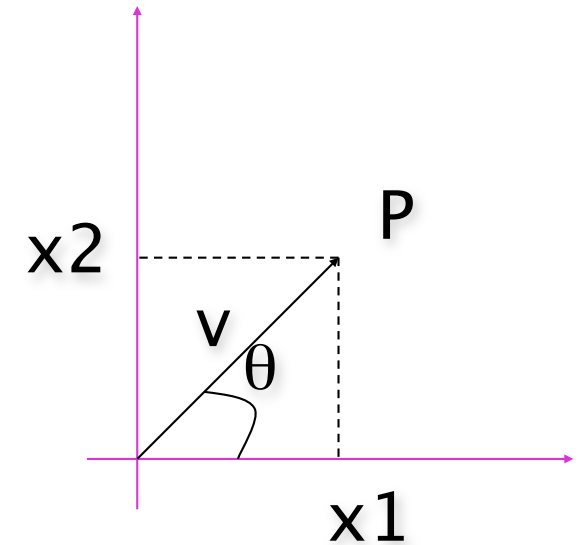
## References:

- Any book on linear algebra!
- [HZ] - chapters 2, 4

Some of the slides in this lecture are courtesy to Prof. Octavia I. Camps, Penn State University

# Vectors (i.e., 2D vectors)

$$\mathbf{v} = (x_1, x_2)$$



Magnitude:  $\|\mathbf{v}\| = \sqrt{x_1^2 + x_2^2}$

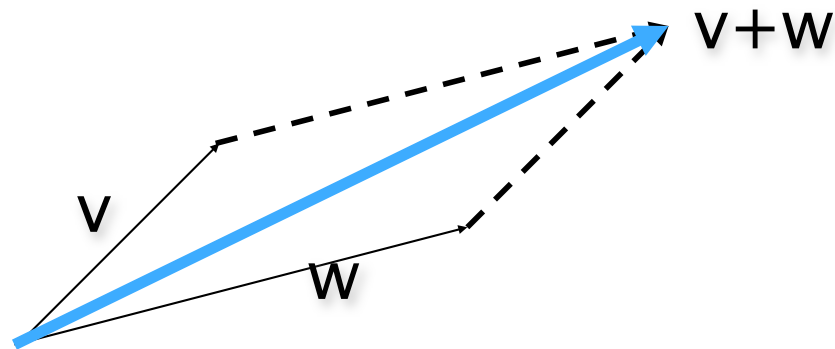
If  $\|\mathbf{v}\| = 1$ ,  $\mathbf{v}$  is a UNIT vector

$$\frac{\mathbf{v}}{\|\mathbf{v}\|} = \left( \frac{x_1}{\|\mathbf{v}\|}, \frac{x_2}{\|\mathbf{v}\|} \right) \text{ is a unit vector}$$

Orientation:  $\theta = \tan^{-1}\left(\frac{x_2}{x_1}\right)$

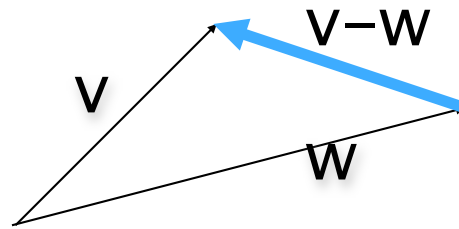
# Vector Addition

$$\mathbf{v} + \mathbf{w} = (x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$$



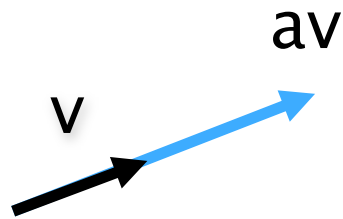
# Vector Subtraction

$$\mathbf{v} - \mathbf{w} = (x_1, x_2) - (y_1, y_2) = (x_1 - y_1, x_2 - y_2)$$

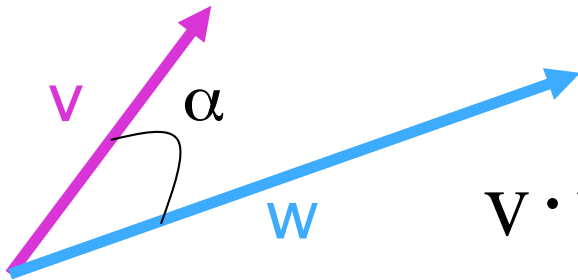


# Scalar Product

$$a\mathbf{v} = a(x_1, x_2) = (ax_1, ax_2)$$



# Inner (dot) Product



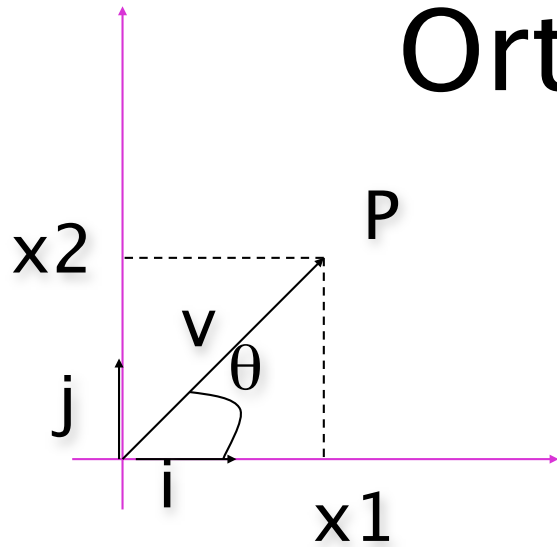
$$\mathbf{v} \cdot \mathbf{w} = (x_1, x_2) \cdot (y_1, y_2) = x_1 y_1 + x_2 y_2$$

The inner product is a **SCALAR!**

$$\mathbf{v} \cdot \mathbf{w} = (x_1, x_2) \cdot (y_1, y_2) = \|\mathbf{v}\| \cdot \|\mathbf{w}\| \cos \alpha$$

$$\text{if } \mathbf{v} \perp \mathbf{w}, \quad \mathbf{v} \cdot \mathbf{w} = ? = 0$$

# Orthonormal Basis



$$\mathbf{i} = (1,0) \quad \|\mathbf{i}\| = 1 \quad \mathbf{i} \cdot \mathbf{j} = 0$$
$$\mathbf{j} = (0,1) \quad \|\mathbf{j}\| = 1$$

$$\mathbf{v} = (x_1, x_2)$$

$$\mathbf{v} = x_1 \mathbf{i} + x_2 \mathbf{j}$$

$$\mathbf{v} \cdot \mathbf{i} = ? = (x_1 \mathbf{i} + x_2 \mathbf{j}) \cdot \mathbf{i} = x_1 \cdot 1 + x_2 \cdot 0 = x_1$$

$$\mathbf{v} \cdot \mathbf{j} = (x_1 \mathbf{i} + x_2 \mathbf{j}) \cdot \mathbf{j} = x_1 \cdot 0 + x_2 \cdot 1 = x_2$$

# Matrices

$$A_{n \times m} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{bmatrix}$$



Pixel's intensity value

$$\text{Sum: } C_{n \times m} = A_{n \times m} + B_{n \times m} \quad c_{ij} = a_{ij} + b_{ij}$$

A and B must have the same dimensions!

$$\text{Example: } \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 8 & 7 \\ 4 & 6 \end{bmatrix}$$



# Matrices

$$A_{n \times m} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{bmatrix} \mathbf{a}_i$$
$$B_{m \times p} = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1p} \\ b_{21} & b_{22} & \dots & b_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ b_{m1} & b_{m2} & \dots & b_{mp} \end{bmatrix} \mathbf{b}_j$$

Product:

$$C_{n \times p} = A_{n \times m} B_{m \times p}$$

$$c_{ij} = \mathbf{a}_i \cdot \mathbf{b}_j = \sum_{k=1}^m a_{ik} b_{kj}$$

A and B must have compatible dimensions!

$$A_{n \times n} B_{n \times n} \neq B_{n \times n} A_{n \times n}$$

# Matrix Inverse

Does not exist for all matrices, necessary (but not sufficient) that the matrix is square

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

$$\mathbf{A}^{-1} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}, \det \mathbf{A} \neq 0$$

If  $\det \mathbf{A} = 0$ ,  $\mathbf{A}$  does not have an inverse.

# Matrix Determinant

Useful value computed from the elements of a *square* matrix **A**

$$\det [a_{11}] = a_{11}$$

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ - a_{13}a_{22}a_{31} - a_{23}a_{32}a_{11} - a_{33}a_{12}a_{21}$$

# Matrix Transpose

Definition:

$$\mathbf{C}_{m \times n} = \mathbf{A}_{n \times m}^T$$
$$c_{ij} = a_{ji}$$

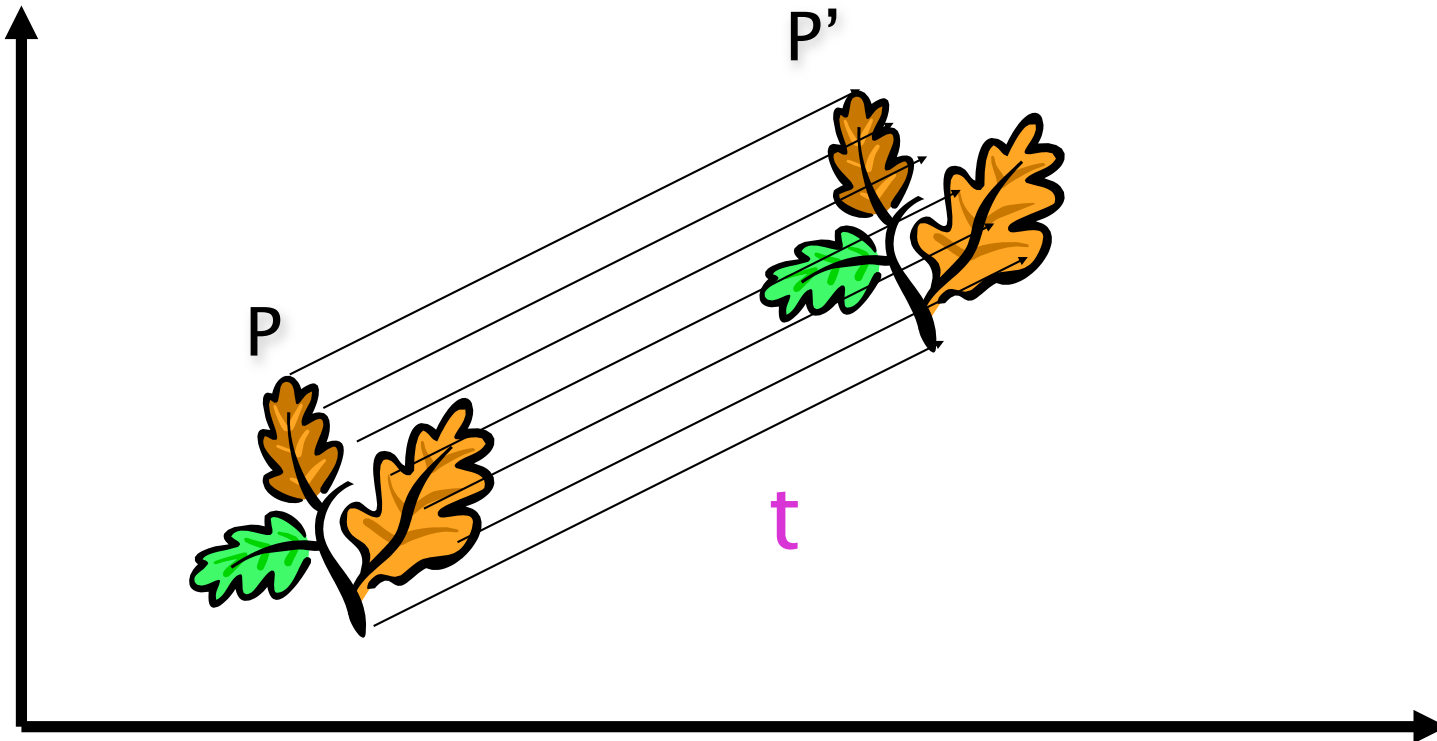
Identities:

$$(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$$
$$(\mathbf{AB})^T = \mathbf{B}^T \mathbf{A}^T$$

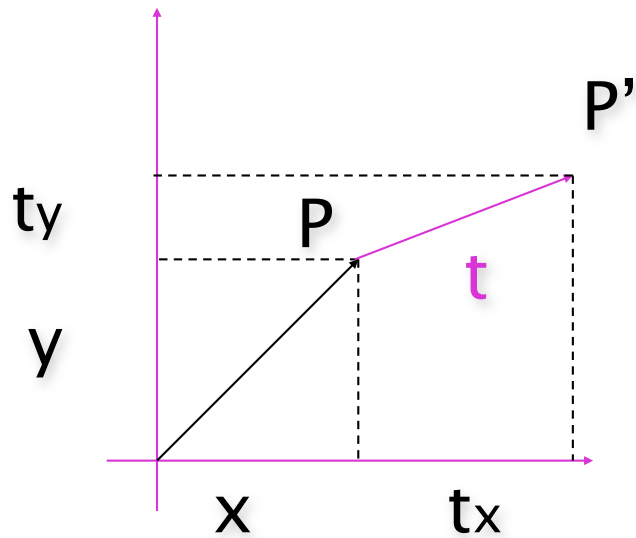
If  $\mathbf{A} = \mathbf{A}^T$ , then  $\mathbf{A}$  is *symmetric*

# 2D Geometrical Transformations

# 2D Translation



# 2D Translation Equation

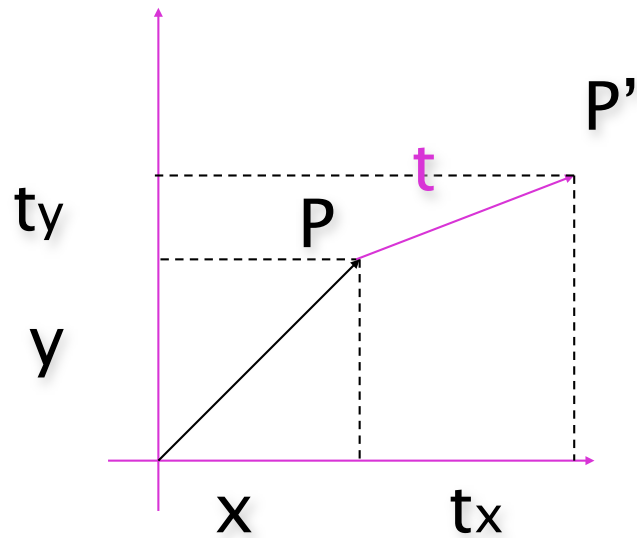


$$\mathbf{P} = (x, y)$$

$$\mathbf{t} = (t_x, t_y)$$

$$\mathbf{P}' = \mathbf{P} + \mathbf{t} = (x + t_x, y + t_y)$$

# 2D Translation using Matrices



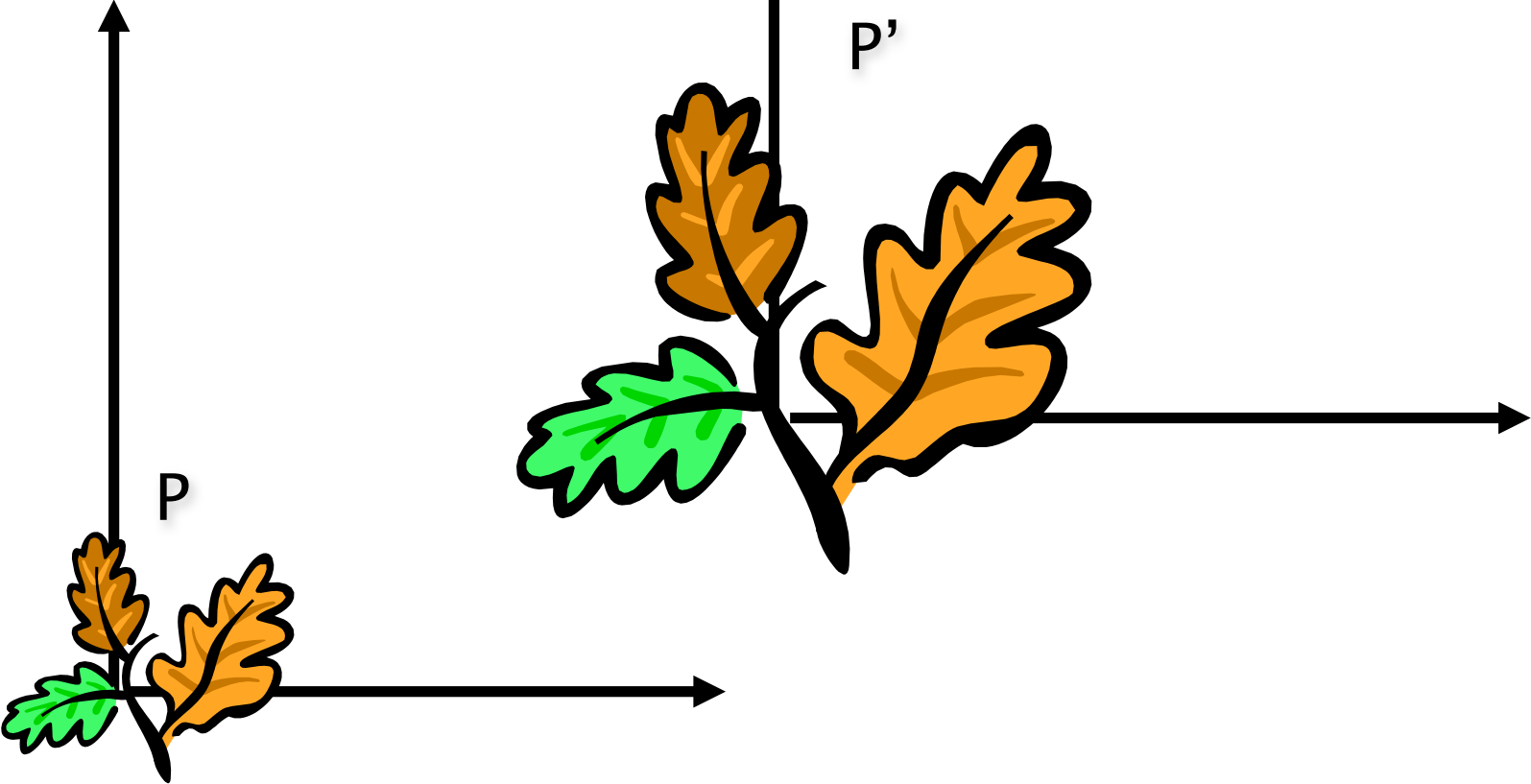
$$\mathbf{P} = (x, y)$$

$$\mathbf{t} = (t_x, t_y)$$

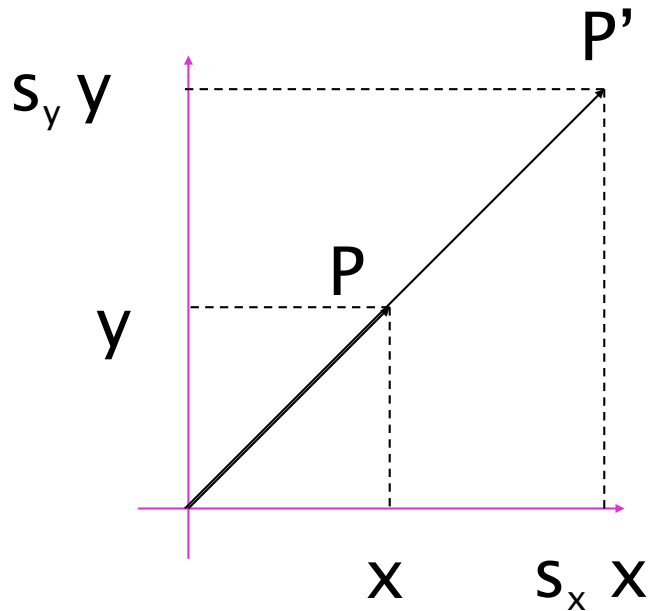
$$\mathbf{P}' \rightarrow \begin{bmatrix} x + t_x \\ y + t_y \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$



# Scaling



# Scaling Equation



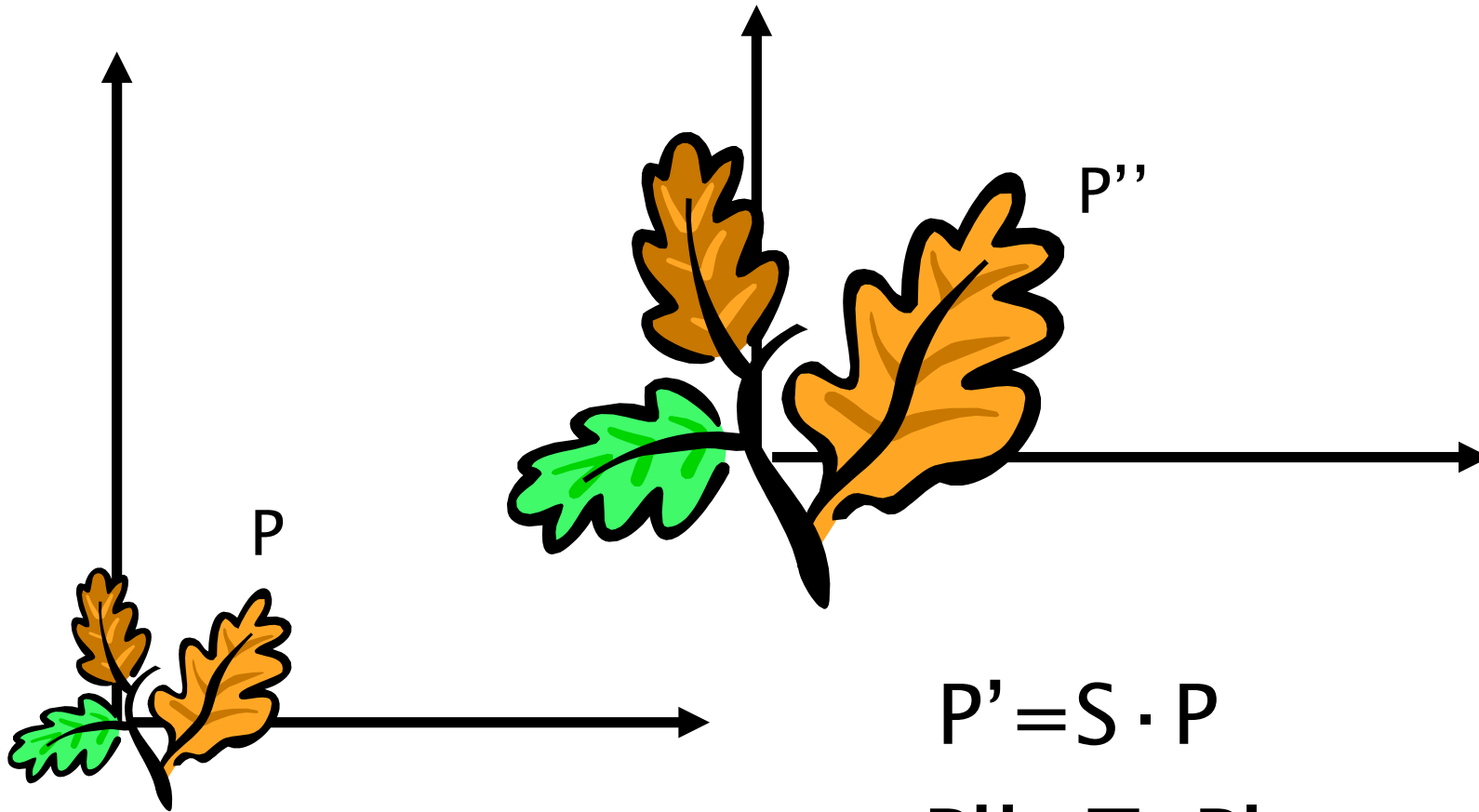
$$\mathbf{P} = (x, y) \rightarrow \mathbf{P}' = (s_x x, s_y y)$$

$$\mathbf{P} = (x, y) \rightarrow (x, y, 1)$$

$$\mathbf{P}' = (s_x x, s_y y) \rightarrow (s_x x, s_y y, 1)$$

$$\mathbf{P}' \rightarrow \begin{bmatrix} s_x x \\ s_y y \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{S}} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{S}' & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \cdot \mathbf{P} = \mathbf{S} \cdot \mathbf{P}$$

# Scaling & Translating



$$P' = S \cdot P$$

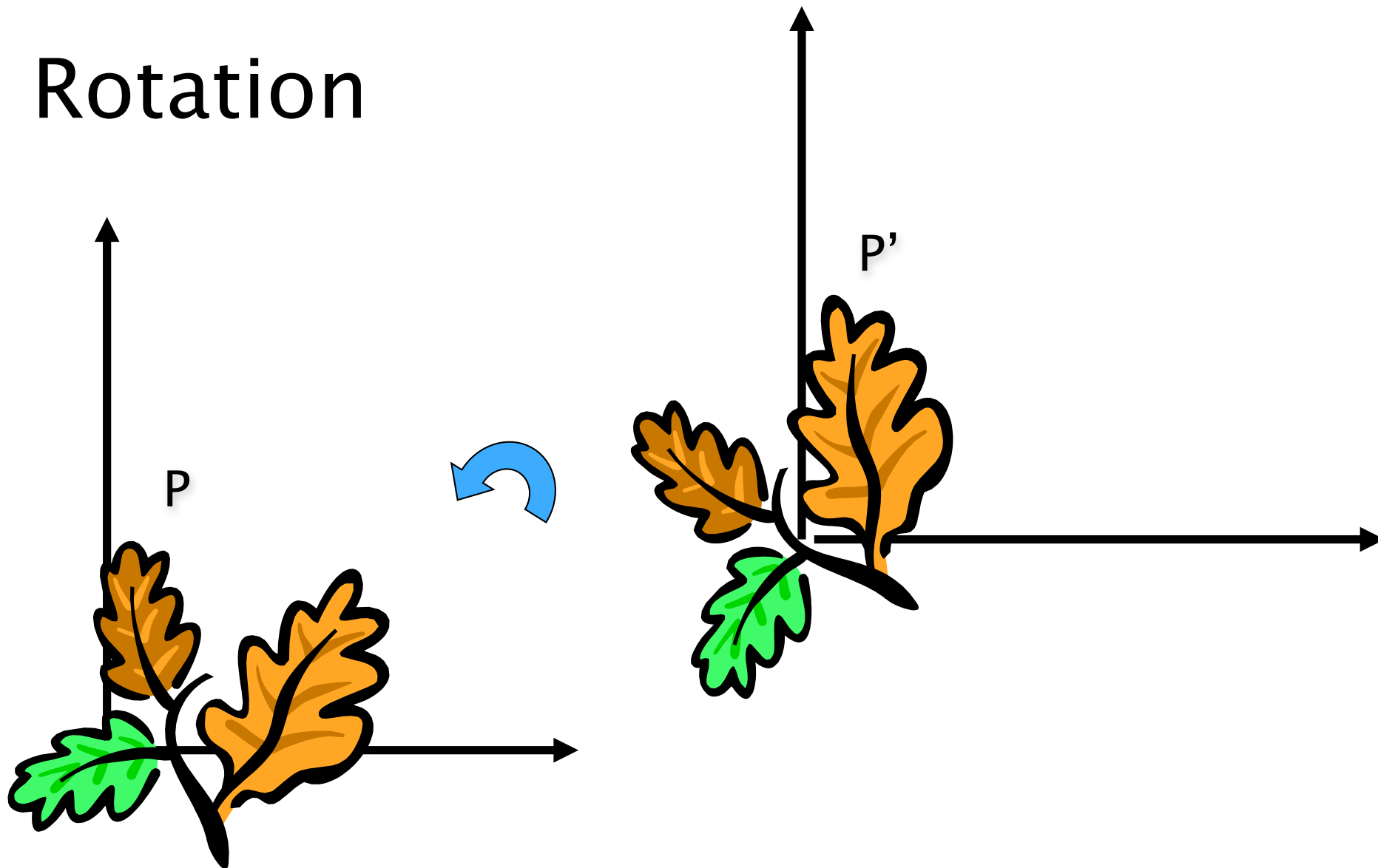
$$P'' = T \cdot P'$$

$$P'' = T \cdot P' = T \cdot (S \cdot P) = (T \cdot S) \cdot P = A \cdot P$$

# Scaling & Translating

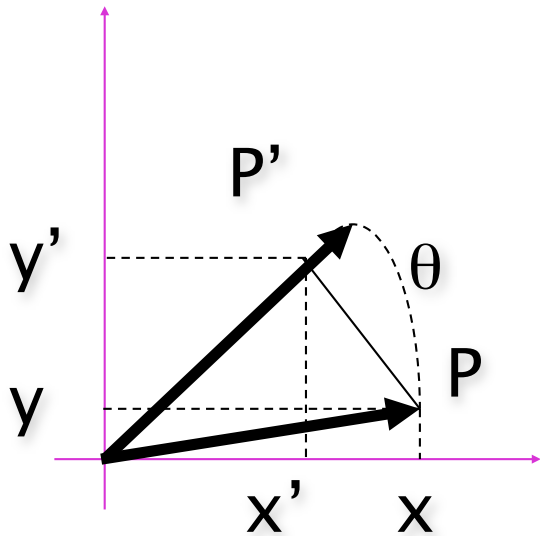
$$\begin{aligned} \mathbf{P}'' = \mathbf{T} \cdot \mathbf{S} \cdot \mathbf{P} &= \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \\ &= \underbrace{\begin{bmatrix} s_x & 0 & t_x \\ 0 & s_y & t_y \\ 0 & 0 & 1 \end{bmatrix}}_A \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} S & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x x + t_x \\ s_y y + t_y \\ 1 \end{bmatrix} \end{aligned}$$

# Rotation



# Rotation Equations

Counter-clockwise rotation by an angle  $\theta$



$$x' = \cos \theta x - \sin \theta y$$

$$y' = \cos \theta y + \sin \theta x$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{P}' = \mathbf{R} \mathbf{P}$$

# Rotation+ Scaling + Translation

$$P' = (T R S) P$$

$$P' = T \cdot R \cdot S \cdot P = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} =$$

$$= \begin{bmatrix} \cos \theta & -\sin \theta & t_x \\ \sin \theta & \cos \theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} =$$

$$= \begin{bmatrix} R' & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} S & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} R' S & t \\ 0 & 1 \end{bmatrix}} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

If  $s_x = s_y$ , this is a  
similarity  
transformation!

# Eigenvalues and Eigenvectors

A eigenvalue  $\lambda$  and eigenvector  $\mathbf{u}$  satisfies

$$\mathbf{A}\mathbf{u} = \lambda\mathbf{u}$$

where  $\mathbf{A}$  is a square matrix.

- ▶ Multiplying  $\mathbf{u}$  by  $\mathbf{A}$  scales  $\mathbf{u}$  by  $\lambda$

Please see geometric demos at:

<http://www.sineofthetimes.org/eigenvectors-of-2-x-2-matrices-a-geometric-exploration/>

See also geometry of covariance matrix:

<http://www.visiondummy.com/2014/04/geometric-interpretation-covariance-matrix/>



# Eigenvalues and Eigenvectors

Rearranging the previous equation gives the system

$$\mathbf{A}\mathbf{u} - \lambda\mathbf{u} = (\mathbf{A} - \lambda\mathbf{I})\mathbf{u} = 0$$

which has a solution if and only if  $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ .

- ▶ The eigenvalues are the roots of this determinant which is polynomial in  $\lambda$ .
- ▶ Substitute the resulting eigenvalues back into  $\mathbf{A}\mathbf{u} = \lambda\mathbf{u}$  and solve to obtain the corresponding eigenvector.