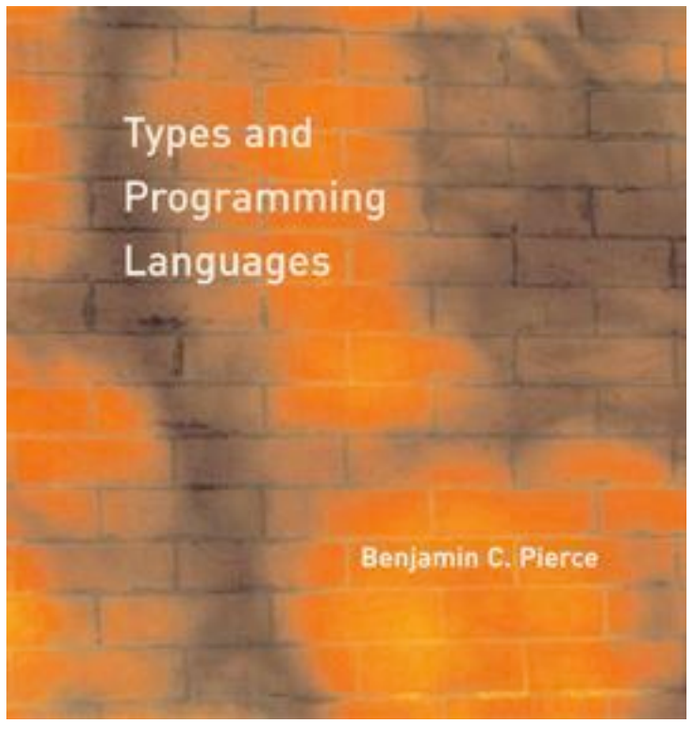
Programming Languages
Fall 2013



Lecture 3: Induction

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Recursive Data Types (trees)

```
data Ast = ANum Integer
         | APlus Ast Ast
         | ATimes Ast Ast

eval (ANum x) = x
eval (ATimes x y) = (eval x) * (eval y)
eval (APlus x y) = (eval x) + (eval y)
```

```
Prelude> eval (ATimes (APlus (ANum 5) (ANum 6)) (ANum 7))
77

-- this tree corresponds to the expression ((5+6)*7)
```

HW2: lexer v1

```
*Main> mylex "(( 12 +340 )*  )"
[LParen,LParen,Number 12,Plus,Number 340,RParen,Times,RParen]
```

```
mylex' :: [Char] -> [Token]
mylex' [] = []
mylex' (x:xs)
    | x `elem` ['0'..'9']  = (Number (digitToInt x)):(mylex' xs)
    | x == '('             = LParen:(mylex' xs)
    | x == ')'             = RParen:(mylex' xs)
    | x == '+'             = Plus:(mylex' xs)
    | x == '-'             = Minus:(mylex' xs)
    | x == '*'             = Times:(mylex' xs)
    | x == ' '             = mylex' xs
    | otherwise            = error "Bad"

mylex2 :: [Char] -> Int -> Bool -> [Token]
mylex2 [] _ False = []
mylex2 [] x True = [Number x]
mylex2 ((Number y):xs) d state = mylex2 xs (d*10+y) True
mylex2 (x:xs) d True = (Number d):x:(mylex2 xs 0 False)
mylex2 (x:xs) d False = x:(mylex2 xs 0 False)

mylex :: [Char] -> [Token]
mylex xs = mylex2 (mylex' xs) 0 False
```

HW2: lexer v2

```
*Main> mylex "(( 12 +340 )*  )"
[LParen,LParen,Number 12,Plus,Number 340,RParen,Times,RParen]
```

```
mylex' :: [Char] -> [Token]
mylex' [] = []
mylex' (x:xs)
    | x `elem` ['0'..'9']  = (Number (digitToInt x)):(mylex' xs)
    | isOperator x         = (operatorToToken x):(mylex' xs)
    | x == ' '             = mylex' xs
    | otherwise            = error "Bad"

isOperator x = x `elem` ['+', '-', '*', '(', ')']

operatorToToken '*' = Times
operatorToToken '-' = Minus
operatorToToken '+' = Plus
operatorToToken '(' = LParen
operatorToToken ')' = RParen

...
```

HW2: parser (assuming correct input)

```
*Main> parse [LParen, Number 3, Times, LParen, Number 5, Plus, Number 6,
RParen, RParen, Times]
(ATimes (ANum 3) (APlus (ANum 5) (ANum 6)),[Times])

*Main> calc "((1+2)*3)"
9
```

```
parse :: [Token] -> (Ast, [Token])
parse ((Number y):xs) = (ANum y, xs)
parse (LParen:xs) = ((opToNode op) left right, remainder)
    where (left, op:rest) = parse xs
          (right, RParen:remainder) = parse rest

opToNode Times = ATimes
opToNode Plus  = APlus
opToNode Minus = AMinus

calc :: [Char] -> Int
calc = eval . fst . parse . mylex
```

Going Meta...

The functional programming style used in OCaml is based on treating programs as data — i.e., on writing functions that manipulate other functions as their inputs and outputs.

Everything in this course is based on treating programs as mathematical objects — i.e., we will be building mathematical theories whose basic objects of study are programs (and whole programming languages).

Jargon: We will be studying the metatheory of programming languages.

Warning!

The material in the next couple of lectures is more slippery than it may first appear.

"I believe it when I hear it" is not a sufficient test of understanding.

A much better test is "I can explain it so that someone else believes it."

Basics of Induction (Review)

Induction

Principle of ordinary induction on natural numbers

Suppose that $\mathbf{P}$ is a predicate on the natural numbers. Then:

If $\mathbf{P}(0)$

and, for all $\mathbf{i}$, $\mathbf{P}(\mathbf{i})$ implies $\mathbf{P}(\mathbf{i}+1)$,

then $\mathbf{P}(\mathbf{n})$ holds for all $\mathbf{n}$.

Example

Theorem: $2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1$, for every n.

Proof:

- Let $P(i)$ be "$2^0 + 2^1 + \ldots + 2^i = 2^{i+1} - 1$."
- Show $P(0)$:

$$2^0 = 1 = 2^1 - 1$$

- Show that $P(i)$ implies $P(i+1)$:

$$\begin{aligned} 2^0 + 2^1 + \ldots + 2^{i+1} &= (2^0 + 2^1 + \ldots + 2^i) + 2^{i+1} \\ &= (2^{i+1} - 1) + 2^{i+1} && \text{by IH} \\ &= 2 \cdot (2^{i+1}) - 1 \\ &= 2^{i+2} - 1 \end{aligned}$$

- The result ($P(n)$ for all n) follows by the principle of induction.

Shorthand form

Theorem: $2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1$, for every n.

Proof: By induction on n.

- Base case ($n = 0$):

$$2^0 = 1 = 2^1 - 1$$

- Inductive case ($n = i + 1$):

$$\begin{aligned} 2^0 + 2^1 + \ldots + 2^{i+1} &= (2^0 + 2^1 + \ldots + 2^i) + 2^{i+1} \\ &= (2^{i+1} - 1) + 2^{i+1} && \text{IH} \\ &= 2 \cdot (2^{i+1}) - 1 \\ &= 2^{i+2} - 1 \end{aligned}$$

Complete Induction

Principle of complete induction on natural numbers

Suppose that $\mathbf{P}$ is a predicate on the natural numbers. Then:
If, for each natural number $\mathbf{n}$,
given $\mathbf{P(i)}$ for all $\mathbf{i < n}$
we can show $\mathbf{P(n)}$,
then $\mathbf{P(n)}$ holds for all $\mathbf{n}$.

Principle of ordinary induction on natural numbers

Suppose that $\mathbf{P}$ is a predicate on the natural numbers. Then:
If $\mathbf{P(0)}$
and, for all $\mathbf{i}$, $\mathbf{P(i)}$ implies $\mathbf{P(i+1)}$,
then $\mathbf{P(n)}$ holds for all $\mathbf{n}$.

Ordinary and complete induction are interderivable — assuming one, we can prove the other.

why?

Thus, the choice of which to use for a particular proof is purely a question of style.

We'll see some other (equivalent) styles as we go along.

Syntax

Simple Arithmetic Expressions

Here is a BNF grammar for a very simple language of arithmetic expressions:

t ::=		terms
	`true`	constant true
	`false`	constant false
	`if t then t else t`	conditional
	`0`	constant zero
	`succ t`	successor
	`pred t`	predecessor
	`iszero t`	zero test

Terminology:

- `t` here is a metavariable

Abstract vs. concrete syntax

Q1: Does this grammar define a set of character strings, a set of token lists, or a set of abstract syntax trees?

Abstract vs. concrete syntax

Q1: Does this grammar define a set of character strings, a set of token lists, or a set of abstract syntax trees?

A: In a sense, all three. But we are primarily interested, here, in abstract syntax trees.

For this reason, grammars like the one on the previous slide are sometimes called abstract grammars. An abstract grammar defines a set of abstract syntax trees and suggests a mapping from character strings to trees.

We then write terms as linear character strings rather than trees simply for convenience. If there is any potential confusion about what tree is intended, we use parentheses to disambiguate.

Q: So, are

```
succ 0
succ (0)
(((succ (((((0)))))))))
```

"the same term"?

What about

```
succ 0
pred (succ (succ 0))
```

?

A more explicit form of the definition

The set $\mathcal{T}$ of terms is the smallest set such that

1. $\{\texttt{true}, \texttt{false}, \texttt{0}\} \subseteq \mathcal{T}$;
2. if $\texttt{t}_1 \in \mathcal{T}$, then $\{\texttt{succ t}_1, \texttt{pred t}_1, \texttt{iszero t}_1\} \subseteq \mathcal{T}$;
3. if $\texttt{t}_1 \in \mathcal{T}$, $\texttt{t}_2 \in \mathcal{T}$, and $\texttt{t}_3 \in \mathcal{T}$, then $\texttt{if t}_1 \texttt{ then t}_2 \texttt{ else t}_3 \in \mathcal{T}$.

Inference rules

An alternate notation for the same definition:

$$\frac{}{\texttt{true} \in \mathcal{T}} \qquad \frac{}{\texttt{false} \in \mathcal{T}} \qquad \frac{}{0 \in \mathcal{T}}$$

$$\frac{t_1 \in \mathcal{T}}{\texttt{succ } t_1 \in \mathcal{T}} \qquad \frac{t_1 \in \mathcal{T}}{\texttt{pred } t_1 \in \mathcal{T}} \qquad \frac{t_1 \in \mathcal{T}}{\texttt{iszero } t_1 \in \mathcal{T}}$$

$$\frac{t_1 \in \mathcal{T} \quad t_2 \in \mathcal{T} \quad t_3 \in \mathcal{T}}{\texttt{if } t_1 \texttt{ then } t_2 \texttt{ else } t_3 \in \mathcal{T}}$$

Note that "the smallest set closed under..." is implied (but often not stated explicitly).

Terminology:

- axiom vs. rule
- concrete rule vs. rule scheme

Terms, concretely

Define an infinite sequence of sets, $\mathcal{S}_0$, $\mathcal{S}_1$, $\mathcal{S}_2$, ..., as follows:

$$
\begin{array}{lcl}
\mathcal{S}_i & = & \emptyset \\
\mathcal{S}_{i+1} & = & \{\texttt{true}, \texttt{false}, 0\} \\
& \cup & \{\texttt{succ } t_1, \texttt{pred } t_1, \texttt{iszero } t_1 \mid t_1 \in \mathcal{S}_i\} \\
& \cup & \{\texttt{if } t_1 \texttt{ then } t_2 \texttt{ else } t_3 \mid t_1, t_2, t_3 \in \mathcal{S}_i\}
\end{array}
$$

Now let

$$\mathcal{S} = \bigcup_i \mathcal{S}_i$$

Comparing the definitions

We have seen two different presentations of terms:

1. as the smallest set that is closed under certain rules ($\mathcal{T}$)
   - explicit inductive definition
   - BNF shorthand
   - inference rule shorthand
2. as the limit ($\mathcal{S}$) of a series of sets (of larger and larger terms)

Comparing the definitions

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   - explicit inductive definition
   - BNF shorthand
   - inference rule shorthand
2. as the limit ($\mathcal{S}$) of a series of sets (of larger and larger terms)

What does it mean to assert that "these presentations are equivalent"?

prove it! (HW3)

Induction on Syntax

Why two definitions?

The two ways of defining the set of terms are both useful:

1. the definition of terms as the smallest set with a certain closure property is compact and easy to read
2. the definition of the set of terms as the limit of a sequence gives us an induction principle for proving things about terms...

Induction on Terms

Definition: The depth of a term t is the smallest i such that $t \in \mathcal{S}_i$.

From the definition of $\mathcal{S}$, it is clear that, if a term t is in $\mathcal{S}_i$, then all of its immediate subterms must be in $\mathcal{S}_{i-1}$, i.e., they must have strictly smaller depths.

This observation justifies the principle of induction on terms.

Let P be a predicate on terms.

> If, for each term s,
>
> given $P(r)$ for all immediate subterms r of s
>
> we can show $P(s)$,
>
> then $P(t)$ holds for all t.

Inductive Function Definitions

The set of constants appearing in a term t, written $\mathit{Consts}(t)$, is defined as follows:

$$
\begin{array}{lcl}
\mathit{Consts}(\texttt{true}) & = & \{\texttt{true}\} \\
\mathit{Consts}(\texttt{false}) & = & \{\texttt{false}\} \\
\mathit{Consts}(0) & = & \{0\} \\
\mathit{Consts}(\texttt{succ}\ t_1) & = & \mathit{Consts}(t_1) \\
\mathit{Consts}(\texttt{pred}\ t_1) & = & \mathit{Consts}(t_1) \\
\mathit{Consts}(\texttt{iszero}\ t_1) & = & \mathit{Consts}(t_1) \\
\mathit{Consts}(\texttt{if}\ t_1\ \texttt{then}\ t_2\ \texttt{else}\ t_3) & = & \mathit{Consts}(t_1) \cup \mathit{Consts}(t_2) \cup \mathit{Consts}(t_3)
\end{array}
$$

Simple, right?

First question:

Normally, a "definition" just assigns a convenient name to a previously-known thing. But here, the "thing" on the right-hand side involves the very name that we are "defining"!

So in what sense is this a definition??

Second question: Suppose we had written this instead...

The set of constants appearing in a term `t`, written BadConsts(`t`), is defined as follows:

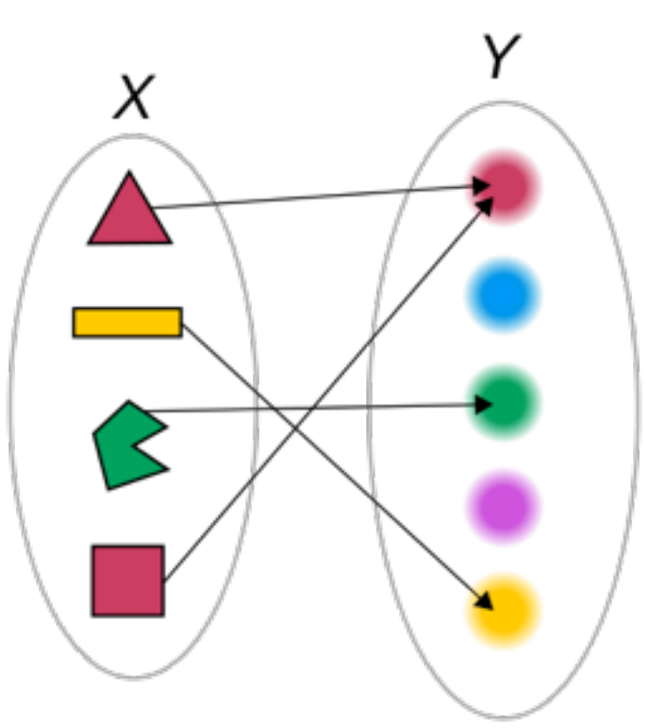
$$
\begin{array}{lcl}
\text{BadConsts}(\texttt{true}) & = & \{\texttt{true}\} \\
\text{BadConsts}(\texttt{false}) & = & \{\texttt{false}\} \\
\text{BadConsts}(\texttt{0}) & = & \{\texttt{0}\} \\
\text{BadConsts}(\texttt{0}) & = & \{\} \\
\text{BadConsts}(\texttt{succ } t_1) & = & \text{BadConsts}(t_1) \\
\text{BadConsts}(\texttt{pred } t_1) & = & \text{BadConsts}(t_1) \\
\text{BadConsts}(\texttt{iszero } t_1) & = & \text{BadConsts}(\texttt{iszero (iszero } t_1\texttt{)})
\end{array}
$$

What is the essential difference between these two definitions? How do we tell the difference between well-formed inductive definitions and ill-formed ones?

What, exactly, does a well-formed inductive definition mean?

First, recall that a function can be viewed as a two-place relation (called the "graph" of the function) with certain properties:

- It is total: every element of its domain occurs at least once in its graph
- It is deterministic: every element of its domain occurs at most once in its graph.



We have seen how to define relations inductively. E.g....

Let Consts be the smallest two-place relation closed under the following rules:

$$(\texttt{true}, \{\texttt{true}\}) \in \textsf{Consts}$$

$$(\texttt{false}, \{\texttt{false}\}) \in \textsf{Consts}$$

$$(0, \{0\}) \in \textsf{Consts}$$

$$\frac{(t_1, C) \in \textsf{Consts}}{(\texttt{succ } t_1, C) \in \textsf{Consts}}$$

$$\frac{(t_1, C) \in \textsf{Consts}}{(\texttt{pred } t_1, C) \in \textsf{Consts}}$$

$$\frac{(t_1, C) \in \textsf{Consts}}{(\texttt{iszero } t_1, C) \in \textsf{Consts}}$$

$$\frac{(t_1, C_1) \in \textsf{Consts} \quad (t_2, C_2) \in \textsf{Consts} \quad (t_3, C_3) \in \textsf{Consts}}{(\texttt{if } t_1 \texttt{ then } t_2 \texttt{ else } t_3, (\textsf{Consts}(t_1) \cup \textsf{Consts}(t_2) \cup \textsf{Consts}(t_3))) \in \textsf{Consts}}$$

This definition certainly defines a relation (i.e., the smallest one with a certain closure property).

Q: How can we be sure that this relation is a function?

This definition certainly defines a relation (i.e., the smallest one with a certain closure property).

Q: How can we be sure that this relation is a function?

A: Prove it!

Theorem: The relation Consts defined by the inference rules a couple of slides ago is total and deterministic.

I.e., for each term t there is exactly one set of terms C such that $(t, C) \in$ Consts.

Proof:

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I.e., for each term t there is exactly one set of terms C such that $(t, C) \in$ Consts.

Proof: By induction on t.

To apply the induction principle for terms, we must show, for an arbitrary term t, that if

> for each immediate subterm s of t, there is exactly one set of terms C_s such that $(s, C_s) \in$ Consts

then

> there is exactly one set of terms C such that $(t, C) \in$ Consts.

Proceed by cases on the form of $\mathtt{t}$.

- If $\mathtt{t}$ is $\mathtt{0}$, $\mathtt{true}$, or $\mathtt{false}$, then we can immediately see from the definition of Consts that there is exactly one set of terms $\mathbf{C}$ (namely $\{\mathtt{t}\}$) such that $(\mathtt{t}, \mathbf{C}) \in$ Consts.

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- If $\mathtt{t}$ is $\mathtt{succ}\ \mathtt{t}_1$, then the induction hypothesis tells us that there is exactly one set of terms $\mathbf{C}_1$ such that $(\mathtt{t}_1, \mathbf{C}_1) \in \mathit{Consts}$. But then it is clear from the definition of Consts that there is exactly one set $\mathbf{C}$ (namely $\mathbf{C}_1$) such that $(\mathtt{t}, \mathbf{C}) \in \mathit{Consts}$.

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- If $\mathtt{t}$ is $\mathtt{succ}\ \mathtt{t}_1$, then the induction hypothesis tells us that there is exactly one set of terms $\mathbf{C}_1$ such that $(\mathtt{t}_1, \mathbf{C}_1) \in \mathit{Consts}$. But then it is clear from the definition of Consts that there is exactly one set $\mathbf{C}$ (namely $\mathbf{C}_1$) such that $(\mathtt{t}, \mathbf{C}) \in \mathit{Consts}$.

Similarly when $\mathtt{t}$ is $\mathtt{pred}\ \mathtt{t}_1$ or $\mathtt{iszero}\ \mathtt{t}_1$.

$$\frac{(\mathtt{t}_1, \mathbf{C}) \in \mathit{Consts}}{(\mathtt{succ}\ \mathtt{t}_1, \mathbf{C}) \in \mathit{Consts}}$$

$$\frac{(\mathtt{t}_1, \mathbf{C}) \in \mathit{Consts}}{(\mathtt{pred}\ \mathtt{t}_1, \mathbf{C}) \in \mathit{Consts}}$$

$$\frac{(\mathtt{t}_1, \mathbf{C}) \in \mathit{Consts}}{(\mathtt{iszero}\ \mathtt{t}_1, \mathbf{C}) \in \mathit{Consts}}$$

$$\frac{(\mathtt{t}_1, \mathbf{C}_1) \in \mathit{Consts} \quad (\mathtt{t}_2, \mathbf{C}_2) \in \mathit{Consts} \quad (\mathtt{t}_3, \mathbf{C}_3) \in \mathit{Consts}}{(\mathtt{if}\ \mathtt{t}_1\ \mathtt{then}\ \mathtt{t}_2\ \mathtt{else}\ \mathtt{t}_3, (\mathit{Consts}(\mathtt{t}_1) \cup \mathit{Consts}(\mathtt{t}_2) \cup \mathit{Consts}(\mathtt{t}_3))) \in \mathit{Consts}}$$

- If $\mathtt{t}$ is $\mathtt{if\ s_1\ then\ s_2\ else\ s_3}$, then the induction hypothesis tells us
  - there is exactly one set of terms $\mathbf{C_1}$ such that $(\mathtt{t_1}, \mathbf{C_1}) \in \mathit{Consts}$
  - there is exactly one set of terms $\mathbf{C_2}$ such that $(\mathtt{t_2}, \mathbf{C_3}) \in \mathit{Consts}$
  - there is exactly one set of terms $\mathbf{C_3}$ such that $(\mathtt{t_3}, \mathbf{C_3}) \in \mathit{Consts}$

  But then it is clear from the definition of *Consts* that there is exactly one set $\mathbf{C}$ (namely $\mathbf{C_1} \cup \mathbf{C_2} \cup \mathbf{C_3}$) such that $(\mathtt{t}, \mathbf{C}) \in \mathit{Consts}$.

$$\frac{(\mathtt{t_1}, \mathbf{C}) \in \mathit{Consts}}{(\mathtt{succ\ t_1}, \mathbf{C}) \in \mathit{Consts}}$$

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How about the bad definition?

$$(\texttt{true}, \{\texttt{true}\}) \in \mathsf{BadConsts}$$

$$(\texttt{false}, \{\texttt{false}\}) \in \mathsf{BadConsts}$$

$$(0, \{0\}) \in \mathsf{BadConsts}$$

$$(0, \{\}) \in \mathsf{BadConsts}$$

$$\frac{(\mathtt{t}_1, \mathbf{C}) \in \mathsf{BadConsts}}{(\texttt{succ } \mathtt{t}_1, \mathbf{C}) \in \mathsf{BadConsts}}$$

$$\frac{(\mathtt{t}_1, \mathbf{C}) \in \mathsf{BadConsts}}{(\texttt{pred } \mathtt{t}_1, \mathbf{C}) \in \mathsf{BadConsts}}$$

$$\frac{(\texttt{iszero (iszero } \mathtt{t}_1), \mathbf{C}) \in \mathsf{BadConsts}}{(\texttt{iszero } \mathtt{t}_1, \mathbf{C}) \in \mathsf{BadConsts}}$$

This set of rules defines a perfectly good relation — it's just that this relation does not happen to be a function!

Just for fun, let's calculate some cases of this relation...

- For what values of $\mathbf{C}$ do we have $(\texttt{false}, \mathbf{C}) \in \mathit{Consts}$?
- For what values of $\mathbf{C}$ do we have $(\texttt{succ 0}, \mathbf{C}) \in \mathit{Consts}$?
- For what values of $\mathbf{C}$ do we have $(\texttt{if false then 0 else 0}, \mathbf{C}) \in \mathit{Consts}$?
- For what values of $\mathbf{C}$ do we have $(\texttt{iszero 0}, \mathbf{C}) \in \mathit{Consts}$?

Another Inductive Definition

$$
\begin{array}{lcl}
\mathsf{size}(\texttt{true}) & = & 1 \\
\mathsf{size}(\texttt{false}) & = & 1 \\
\mathsf{size}(\texttt{0}) & = & 1 \\
\mathsf{size}(\texttt{succ } t_1) & = & \mathsf{size}(t_1) + 1 \\
\mathsf{size}(\texttt{pred } t_1) & = & \mathsf{size}(t_1) + 1 \\
\mathsf{size}(\texttt{iszero } t_1) & = & \mathsf{size}(t_1) + 1 \\
\mathsf{size}(\texttt{if } t_1 \texttt{ then } t_2 \texttt{ else } t_3) & = & \mathsf{size}(t_1) + \mathsf{size}(t_2) + \mathsf{size}(t_3) + 1
\end{array}
$$

Another proof by induction

Theorem: The number of distinct constants in a term is at most the size of the term. I.e., $|\text{Consts}(t)| \leq \text{size}(t)$.

Proof:

Another proof by induction

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Proof: By induction on $\mathtt{t}$.

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Assuming the desired property for immediate subterms of t, we must prove it for t itself.

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Proof: By induction on $\mathtt{t}$.

Assuming the desired property for immediate subterms of $\mathtt{t}$, we must prove it for $\mathtt{t}$ itself.

There are three cases to consider:

Case: $\mathtt{t}$ is a constant

Immediate: $|\mathsf{Consts}(\mathtt{t})| = |\{\mathtt{t}\}| = 1 = \mathsf{size}(\mathtt{t})$.

Another proof by induction

Theorem: The number of distinct constants in a term is at most the size of the term. I.e., $|\mathsf{Consts}(\mathtt{t})| \leq \mathsf{size}(\mathtt{t})$.

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There are three cases to consider:

Case: $\mathtt{t}$ is a constant

Immediate: $|\mathsf{Consts}(\mathtt{t})| = |\{\mathtt{t}\}| = 1 = \mathsf{size}(\mathtt{t})$.

Case: $\mathtt{t} = \mathtt{succ}\ \mathtt{t}_1$, $\mathtt{pred}\ \mathtt{t}_1$, or $\mathtt{iszero}\ \mathtt{t}_1$

By the induction hypothesis, $|\mathsf{Consts}(\mathtt{t}_1)| \leq \mathsf{size}(\mathtt{t}_1)$. We now calculate as follows: $|\mathsf{Consts}(\mathtt{t})| = |\mathsf{Consts}(\mathtt{t}_1)| \leq \mathsf{size}(\mathtt{t}_1) < \mathsf{size}(\mathtt{t})$.

Case: $t = \texttt{if } t_1 \texttt{ then } t_2 \texttt{ else } t_3$

By the induction hypothesis, $|\mathsf{Consts}(t_1)| \leq \mathsf{size}(t_1)$, $|\mathsf{Consts}(t_2)| \leq \mathsf{size}(t_2)$, and $|\mathsf{Consts}(t_3)| \leq \mathsf{size}(t_3)$. We now calculate as follows:

$$\begin{aligned}
|\mathsf{Consts}(t)| &= |\mathsf{Consts}(t_1) \cup \mathsf{Consts}(t_2) \cup \mathsf{Consts}(t_3)| \\
&\leq |\mathsf{Consts}(t_1)| + |\mathsf{Consts}(t_2)| + |\mathsf{Consts}(t_3)| \\
&\leq \mathsf{size}(t_1) + \mathsf{size}(t_2) + \mathsf{size}(t_3) \\
&< \mathsf{size}(t).
\end{aligned}$$

Operational Semantics

Abstract Machines

An abstract machine consists of:

- a set of states
- a transition relation on states, written $\longrightarrow$

A state records all the information in the machine at a given moment. For example, an abstract-machine-style description of a conventional microprocessor would include the program counter, the contents of the registers, the contents of main memory, and the machine code program being executed.

For the very simple languages we are considering at the moment, however, the term being evaluated is the whole state of the abstract machine.

Nb. Often, the transition relation is actually a partial function: i.e., from a given state, there is at most one possible next state. But in general there may be many.

Operational semantics for Booleans

Syntax of terms and values

`t` ::=		terms
	`true`	constant true
	`false`	constant false
	`if t then t else t`	conditional
`v` ::=		values
	`true`	true value
	`false`	false value

The evaluation relation $t \longrightarrow t'$ is the smallest relation closed under the following rules:

$$\texttt{if true then } t_2 \texttt{ else } t_3 \longrightarrow t_2 \qquad \text{(E-IfTrue)}$$

$$\texttt{if false then } t_2 \texttt{ else } t_3 \longrightarrow t_3 \qquad \text{(E-IfFalse)}$$

$$\frac{t_1 \longrightarrow t_1'}{\texttt{if } t_1 \texttt{ then } t_2 \texttt{ else } t_3 \longrightarrow \texttt{if } t_1' \texttt{ then } t_2 \texttt{ else } t_3} \qquad \text{(E-If)}$$

Terminology

Computation rules:

$$\texttt{if true then } t_2 \texttt{ else } t_3 \longrightarrow t_2 \qquad \text{(E-IfTrue)}$$

$$\texttt{if false then } t_2 \texttt{ else } t_3 \longrightarrow t_3 \qquad \text{(E-IfFalse)}$$

Congruence rule:

$$\frac{t_1 \longrightarrow t_1'}{\texttt{if } t_1 \texttt{ then } t_2 \texttt{ else } t_3 \longrightarrow \texttt{if } t_1' \texttt{ then } t_2 \texttt{ else } t_3} \qquad \text{(E-If)}$$

Computation rules perform "real" computation steps.

Congruence rules determine where computation rules can be applied next.

Syntax/Semantics of Untyped Booleans

𝔹 *(untyped)*

Syntax

t ::=		*terms:*
	`true`	*constant true*
	`false`	*constant false*
	`if t then t else t`	*conditional*
v ::=		*values:*
	`true`	*true value*
	`false`	*false value*

Evaluation $t \longrightarrow t'$

$$\texttt{if true then } t_2 \texttt{ else } t_3 \longrightarrow t_2 \qquad \text{(E-IfTrue)}$$

$$\texttt{if false then } t_2 \texttt{ else } t_3 \longrightarrow t_3 \qquad \text{(E-IfFalse)}$$

$$\frac{t_1 \longrightarrow t_1'}{\texttt{if } t_1 \texttt{ then } t_2 \texttt{ else } t_3 \longrightarrow \texttt{if } t_1' \texttt{ then } t_2 \texttt{ else } t_3} \qquad \text{(E-If)}$$

Figure 3-1: Booleans (B)

Evaluation, more explicitly

$\longrightarrow$ is the smallest two-place relation closed under the following rules:

$$((\texttt{if true then } t_2 \texttt{ else } t_3), t_2) \quad \in \quad \longrightarrow$$

$$((\texttt{if false then } t_2 \texttt{ else } t_3), t_3) \quad \in \quad \longrightarrow$$

$$\frac{(t_1, t_1') \quad \in \quad \longrightarrow}{((\texttt{if } t_1 \texttt{ then } t_2 \texttt{ else } t_3), (\texttt{if } t_1' \texttt{ then } t_2 \texttt{ else } t_3)) \quad \in \quad \longrightarrow}$$

Digression

Suppose we wanted to change our evaluation strategy so that the `then` and `else` branches of an `if` get evaluated (in that order) before the guard. How would we need to change the rules?

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Suppose we wanted to change our evaluation strategy so that the `then` and `else` branches of an `if` get evaluated (in that order) before the guard. How would we need to change the rules?

Suppose, moreover, that if the evaluation of the `then` and `else` branches leads to the same value, we want to immediately produce that value ("short-circuiting" the evaluation of the guard). How would we need to change the rules?

Digression

Suppose we wanted to change our evaluation strategy so that the `then` and `else` branches of an `if` get evaluated (in that order) before the guard. How would we need to change the rules?

Suppose, moreover, that if the evaluation of the `then` and `else` branches leads to the same value, we want to immediately produce that value ("short-circuiting" the evaluation of the guard). How would we need to change the rules?

Of the rules we just invented, which are computation rules and which are congruence rules?

Evaluation, more explicitly

$\longrightarrow$ is the smallest two-place relation closed under the following rules:

$$((\texttt{if true then } t_2 \texttt{ else } t_3), t_2) \quad \in \quad \longrightarrow$$

$$((\texttt{if false then } t_2 \texttt{ else } t_3), t_3) \quad \in \quad \longrightarrow$$

$$\frac{(t_1, t_1') \quad \in \quad \longrightarrow}{((\texttt{if } t_1 \texttt{ then } t_2 \texttt{ else } t_3), (\texttt{if } t_1' \texttt{ then } t_2 \texttt{ else } t_3)) \quad \in \quad \longrightarrow}$$

Even more explicitly...

What is the generating function corresponding to these rules?

(exercise)

Reasoning about Evaluation

Derivations

We can record the "justification" for a particular pair of terms that are in the evaluation relation in the form of a tree.

(on the board)

Terminology:

- These trees are called derivation trees (or just derivations)
- The final statement in a derivation is its conclusion
- We say that the derivation is a witness for its conclusion (or a proof of its conclusion) — it records all the reasoning steps that justify the conclusion.

Observation

Lemma: Suppose we are given a derivation tree $\mathcal{D}$ witnessing the pair (t, t') in the evaluation relation. Then either

1. the final rule used in $\mathcal{D}$ is E-IfTrue and we have $t = \texttt{if true then } t_2 \texttt{ else } t_3$ and $t' = t_2$, for some t_2 and t_3, or
2. the final rule used in $\mathcal{D}$ is E-IfFalse and we have $t = \texttt{if false then } t_2 \texttt{ else } t_3$ and $t' = t_3$, for some t_2 and t_3, or
3. the final rule used in $\mathcal{D}$ is E-If and we have $t = \texttt{if } t_1 \texttt{ then } t_2 \texttt{ else } t_3$ and $t' = \texttt{if } t_1' \texttt{ then } t_2 \texttt{ else } t_3$, for some t_1, t_1', t_2, and t_3; moreover, the immediate subderivation of $\mathcal{D}$ witnesses $(t_1, t_1') \in \longrightarrow$.

Induction on Derivations

We can now write proofs about evaluation "by induction on derivation trees."

Given an arbitrary derivation $\mathcal{D}$ with conclusion $t \longrightarrow t'$, we assume the desired result for its immediate sub-derivation (if any) and proceed by a case analysis (using the previous lemma) of the final evaluation rule used in constructing the derivation tree.

E.g....

Induction on Derivations — Example

Theorem: If $t \longrightarrow t'$ — i.e., if $(t, t') \in \longrightarrow$ — then $\mathsf{size}(t) > \mathsf{size}(t')$.

Proof: By induction on a derivation $\mathcal{D}$ of $t \longrightarrow t'$.

1. Suppose the final rule used in $\mathcal{D}$ is E-IfTrue, with $t = \texttt{if true then } t_2 \texttt{ else } t_3$ and $t' = t_2$. Then the result is immediate from the definition of size.
2. Suppose the final rule used in $\mathcal{D}$ is E-IfFalse, with $t = \texttt{if false then } t_2 \texttt{ else } t_3$ and $t' = t_3$. Then the result is again immediate from the definition of size.
3. Suppose the final rule used in $\mathcal{D}$ is E-If, with $t = \texttt{if } t_1 \texttt{ then } t_2 \texttt{ else } t_3$ and $t' = \texttt{if } t_1' \texttt{ then } t_2 \texttt{ else } t_3$, where $(t_1, t_1') \in \longrightarrow$ is witnessed by a derivation $\mathcal{D}_\infty$. By the induction hypothesis, $\mathsf{size}(t_1) > \mathsf{size}(t_1')$. But then, by the definition of size, we have $\mathsf{size}(t) > \mathsf{size}(t')$.

Normal forms

A normal form is a term that cannot be evaluated any further — i.e., a term t is a normal form (or "is in normal form") if there is no t' such that $t \longrightarrow t'$.

A normal form is a state where the abstract machine is halted — i.e., it can be regarded as a "result" of evaluation.

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A normal form is a state where the abstract machine is halted — i.e., it can be regarded as a "result" of evaluation.

Recall that we intended the set of values (the boolean constants `true` and `false`) to be exactly the possible "results of evaluation."

Did we get this definition right?

Values = normal forms

Theorem: A term `t` is a value iff it is in normal form.

Proof:

Values = normal forms

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Proof: The $\Longrightarrow$ direction is immediate from the definition of the evaluation relation.

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For the $\Longleftarrow$ direction,

Values = normal forms

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Proof: The $\Longrightarrow$ direction is immediate from the definition of the evaluation relation.

For the $\Longleftarrow$ direction, it is convenient to prove the contrapositive: If `t` is not a value, then it is not a normal form.

Values = normal forms

Theorem: A term $\mathtt{t}$ is a value iff it is in normal form.

Proof: The $\Longrightarrow$ direction is immediate from the definition of the evaluation relation.

For the $\Longleftarrow$ direction, it is convenient to prove the contrapositive: If $\mathtt{t}$ is not a value, then it is not a normal form. The argument goes by induction on $\mathtt{t}$.

Note, first, that $\mathtt{t}$ must have the form $\mathtt{if\ t_1\ then\ t_2\ else\ t_3}$ (otherwise it would be a value). If $\mathtt{t_1}$ is $\mathtt{true}$ or $\mathtt{false}$, then rule E-IfTrue or E-IfFalse applies to $\mathtt{t}$, and we are done. Otherwise, $\mathtt{t_1}$ is not a value and so, by the induction hypothesis, there is some $\mathtt{t_1'}$ such that $\mathtt{t_1} \longrightarrow \mathtt{t_1'}$. But then rule E-If yields

$$\mathtt{if\ t_1\ then\ t_2\ else\ t_3} \longrightarrow \mathtt{if\ t_1'\ then\ t_2\ else\ t_3}$$

i.e., $\mathtt{t}$ is not in normal form.

Numbers

New syntactic forms

t	::=	...	terms
		0	constant zero
		succ t	successor
		pred t	predecessor
		iszero t	zero test
v	::=	...	values
		nv	numeric value
nv	::=		numeric values
		0	zero value
		succ nv	successor value

New evaluation rules

$t \longrightarrow t'$

$$\frac{t_1 \longrightarrow t_1'}{\texttt{succ}\ t_1 \longrightarrow \texttt{succ}\ t_1'} \qquad \text{(E-Succ)}$$

$$\texttt{pred}\ 0 \longrightarrow 0 \qquad \text{(E-PredZero)}$$

$$\texttt{pred}\ (\texttt{succ}\ nv_1) \longrightarrow nv_1 \qquad \text{(E-PredSucc)}$$

$$\frac{t_1 \longrightarrow t_1'}{\texttt{pred}\ t_1 \longrightarrow \texttt{pred}\ t_1'} \qquad \text{(E-Pred)}$$

$$\texttt{iszero}\ 0 \longrightarrow \texttt{true} \qquad \text{(E-IszeroZero)}$$

$$\texttt{iszero}\ (\texttt{succ}\ nv_1) \longrightarrow \texttt{false} \qquad \text{(E-IszeroSucc)}$$

$$\frac{t_1 \longrightarrow t_1'}{\texttt{iszero}\ t_1 \longrightarrow \texttt{iszero}\ t_1'} \qquad \text{(E-IsZero)}$$

Syntax/Semantics of Arithmetics

𝔹 *(untyped)*

Syntax

t ::=		*terms:*
	`true`	*constant true*
	`false`	*constant false*
	`if t then t else t`	*conditional*
v ::=		*values:*
	`true`	*true value*
	`false`	*false value*

Evaluation $t \longrightarrow t'$

$$\texttt{if true then } t_2 \texttt{ else } t_3 \longrightarrow t_2 \quad \text{(E-IfTrue)}$$

$$\texttt{if false then } t_2 \texttt{ else } t_3 \longrightarrow t_3 \quad \text{(E-IfFalse)}$$

$$\frac{t_1 \longrightarrow t_1'}{\texttt{if } t_1 \texttt{ then } t_2 \texttt{ else } t_3 \longrightarrow \texttt{if } t_1' \texttt{ then } t_2 \texttt{ else } t_3} \quad \text{(E-If)}$$

𝔹 ℕ *(untyped)* — *Extends* **B** *(3-1)*

New syntactic forms

t ::=	...	*terms:*
	`0`	*constant zero*
	`succ t`	*successor*
	`pred t`	*predecessor*
	`iszero t`	*zero test*
v ::=	...	*values:*
	`nv`	*numeric value*
nv ::=		*numeric values:*
	`0`	*zero value*
	`succ nv`	*successor value*

New evaluation rules $t \longrightarrow t'$

$$\frac{t_1 \longrightarrow t_1'}{\texttt{succ } t_1 \longrightarrow \texttt{succ } t_1'} \quad \text{(E-Succ)}$$

$$\texttt{pred 0} \longrightarrow \texttt{0} \quad \text{(E-PredZero)}$$

$$\texttt{pred (succ } nv_1\texttt{)} \longrightarrow nv_1 \quad \text{(E-PredSucc)}$$

$$\frac{t_1 \longrightarrow t_1'}{\texttt{pred } t_1 \longrightarrow \texttt{pred } t_1'} \quad \text{(E-Pred)}$$

$$\texttt{iszero 0} \longrightarrow \texttt{true} \quad \text{(E-IszeroZero)}$$

$$\texttt{iszero (succ } nv_1\texttt{)} \longrightarrow \texttt{false} \quad \text{(E-IszeroSucc)}$$

$$\frac{t_1 \longrightarrow t_1'}{\texttt{iszero } t_1 \longrightarrow \texttt{iszero } t_1'} \quad \text{(E-IsZero)}$$

Figure 3-2: Arithmetic expressions (NB)

Values are normal forms

Our observation a few slides ago that all values are in normal form still holds for the extended language.

Is the converse true? I.e., is every normal form a value?

Stuck terms

Is the converse true? I.e., is every normal form a value?

No: some terms are stuck.

Formally, a stuck term is one that is a normal form but not a value.

Stuck terms model run-time errors.

Multi-step evaluation.

The multi-step evaluation relation, $\longrightarrow^*$, is the reflexive, transitive closure of single-step evaluation.

I.e., it is the smallest relation closed under the following rules:

$$\frac{t \longrightarrow t'}{t \longrightarrow^* t'}$$

$$t \longrightarrow^* t$$

$$\frac{t \longrightarrow^* t' \qquad t' \longrightarrow^* t''}{t \longrightarrow^* t''}$$

Termination of evaluation

Theorem: For every t there is some normal form t' such that $t \longrightarrow^* t'$.

Proof:

Termination of evaluation

Theorem: For every t there is some normal form t' such that $t \longrightarrow^* t'$.

Proof:

- First, recall that single-step evaluation strictly reduces the size of the term:

  if $t \longrightarrow t'$, then $\mathsf{size}(t) > \mathsf{size}(t')$

- Now, assume (for a contradiction) that

  $t_0, t_1, t_2, t_3, t_4, \ldots$

  is an infinite-length sequence such that

  $t_0, \longrightarrow t_1, \longrightarrow t_2, \longrightarrow t_3, \longrightarrow t_4 \longrightarrow \cdots,$

- Then

  $\mathsf{size}(t_0), \mathsf{size}(t_1), \mathsf{size}(t_2), \mathsf{size}(t_3), \mathsf{size}(t_4), \ldots$

  is an infinite, strictly decreasing, sequence of natural numbers.

- But such a sequence cannot exist — contradiction!

Termination Proofs

Most termination proofs have the same basic form:

> **Theorem:** The relation $R \subseteq X \times X$ is terminating — i.e., there are no infinite sequences x_0, x_1, x_2, etc. such that $(x_i, x_{i+1}) \in R$ for each i.
>
> **Proof:**
>
> 1. Choose
>   - a well-founded set $(W, <)$ — i.e., a set W with a partial order $<$ such that there are no infinite descending chains $w_0 > w_1 > w_2 > \ldots$ in W
>   - a function f from X to W
> 2. Show $f(x) > f(y)$ for all $(x, y) \in R$
> 3. Conclude that there are no infinite sequences x_0, x_1, x_2, etc. such that $(x_i, x_{i+1}) \in R$ for each i), since, if there were, we could construct an infinite descending chain in W.

Big-Step Semantics

$$v \Downarrow v \qquad \text{(B-Value)}$$

$$\frac{t_1 \Downarrow \texttt{true} \qquad t_2 \Downarrow v_2}{\texttt{if } t_1 \texttt{ then } t_2 \texttt{ else } t_3 \Downarrow v_2} \qquad \text{(B-IfTrue)}$$

$$\frac{t_1 \Downarrow \texttt{false} \qquad t_3 \Downarrow v_3}{\texttt{if } t_1 \texttt{ then } t_2 \texttt{ else } t_3 \Downarrow v_3} \qquad \text{(B-IfFalse)}$$

$$\frac{t_1 \Downarrow nv_1}{\texttt{succ } t_1 \Downarrow \texttt{succ } nv_1} \qquad \text{(B-Succ)}$$

$$\frac{t_1 \Downarrow 0}{\texttt{pred } t_1 \Downarrow 0} \qquad \text{(B-PredZero)}$$

$$\frac{t_1 \Downarrow \texttt{succ } nv_1}{\texttt{pred } t_1 \Downarrow nv_1} \qquad \text{(B-PredSucc)}$$

$$\frac{t_1 \Downarrow 0}{\texttt{iszero } t_1 \Downarrow \texttt{true}} \qquad \text{(B-IszeroZero)}$$

$$\frac{t_1 \Downarrow \texttt{succ } nv_1}{\texttt{iszero } t_1 \Downarrow \texttt{false}} \qquad \text{(B-IszeroSucc)}$$

Show that the small-step and big-step semantics for this language coincide, i.e. $t \longrightarrow^* v$ iff $t \Downarrow v$. □