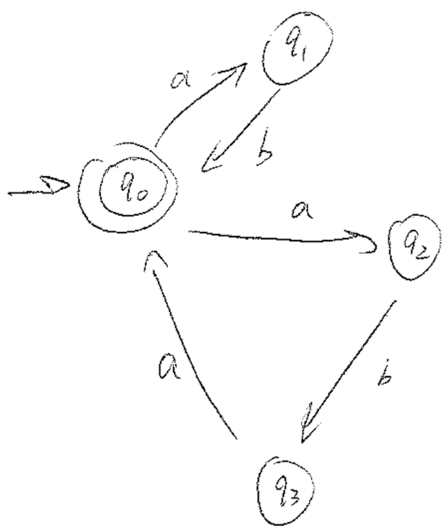
1. Convert the following NFAs from HW3 to DFA:
   (a) {ab, aba}*
   (b) bitstrings with 0 as the third last symbol
   (c) bitstrings that contain 0100

   How do these converted DFAs compare to your own DFAs in HW3?

   Solution:

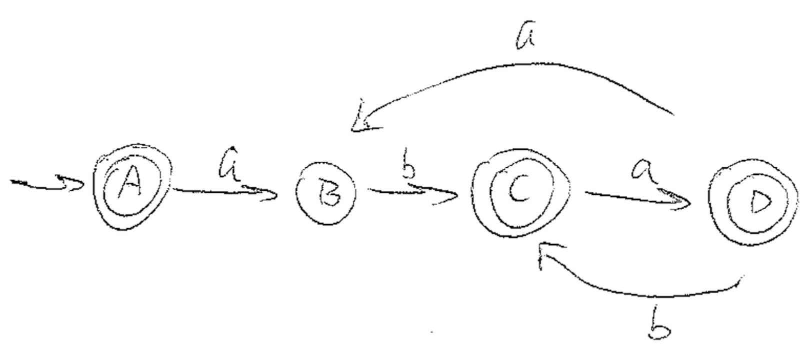
   (a) NFA:



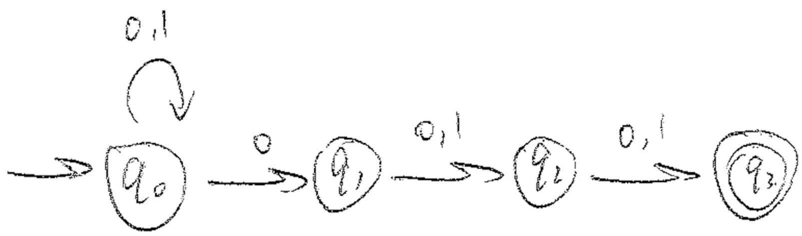
conversion:

	a	b
$A = \{q_0\}$	B	
$B = \{q_1, q_2\}$		C
$C = \{q_0, q_3\}$	D	
$D = \{q_0, q_1, q_2\}$	B	C

DFA:



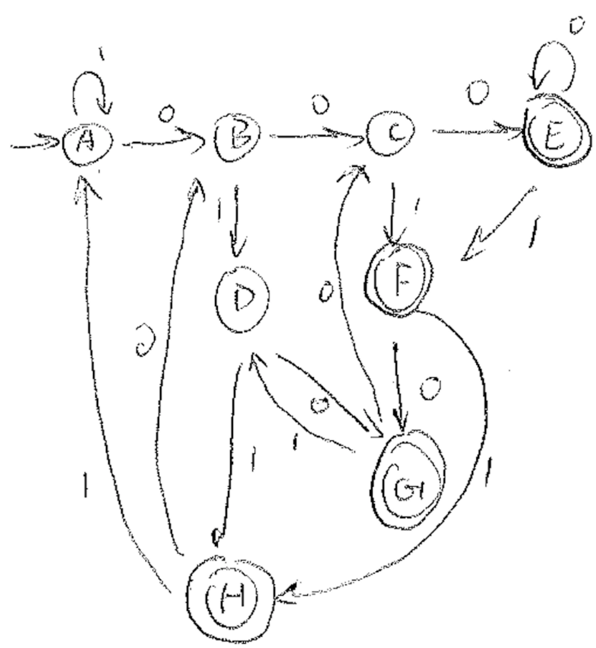
(b) NFA:



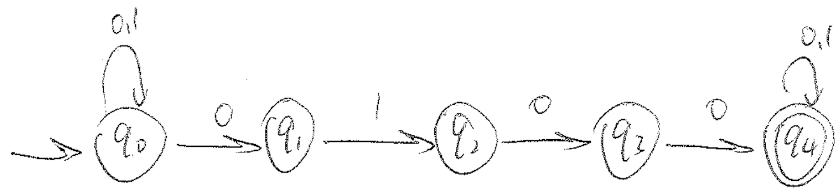
conversion:

	0	1
$A = \{q_0\}$	B	A
$B = \{q_0, q_1\}$	C	D
$C = \{q_0, q_1, q_2\}$	E	F
$D = \{q_0, q_2\}$	G	H
$E = \{q_0, q_1, q_2, q_3\}$	E	F
$F = \{q_0, q_2, q_3\}$	G	H
$G = \{q_0, q_1, q_3\}$	C	D
$H = \{q_0, q_3\}$	B	A

DFA:



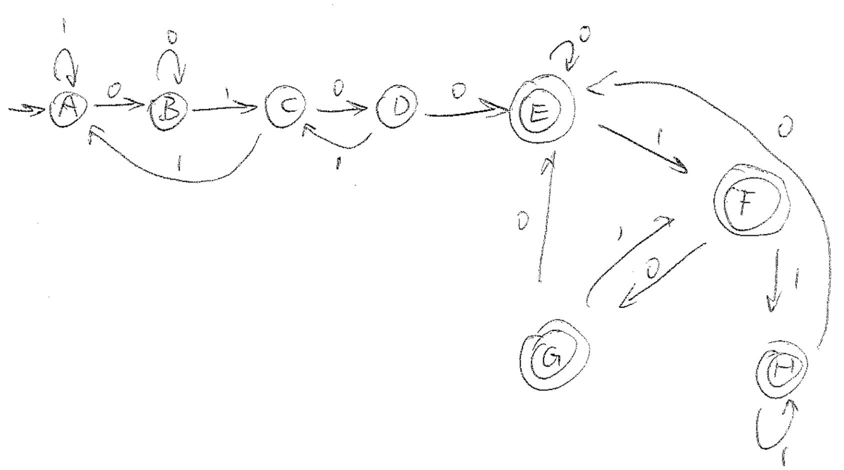
(c) NFA:



conversion:

	0	1
$A = \{q_0\}$	B	A
$B = \{q_0, q_1\}$	B	C
$C = \{q_0, q_2\}$	D	A
$D = \{q_0, q_1, q_3\}$	E	C
$E = \{q_0, q_1, q_4\}$	E	F
$F = \{q_0, q_2, q_4\}$	G	H
$G = \{q_0, q_1, q_3, q_4\}$	E	F
$H = \{q_0, q_4\}$	E	H

DFA:



Hint: this DFA is not likely to be same as your own DFA in HW3; 4 acceptance states could be merged to one.

2. For any given epsilon-free NFA $M = (Q, \Sigma, \delta, q_0, F)$,
   (a) construct a DFA M' such that $L(M) = L(M')$.
   (b) prove that $L(M) = L(M')$.
   i.e., for all $w \in L(M)$, you want to show $w \in L(M')$, and for all $w \in L(M')$, you want to show $w \in L(M)$.
   Hint: first prove a lemma by induction:
   $\forall q \in Q, \forall w \in \Sigma^*, \delta^*(q, w) = \delta'^*(\{q\}, w)$.

Solution:

(also see slides “week3 (NFA) ”, page 12)

(a)

$$M' = (Q', \Sigma, \delta', q_0', F'),$$

where

$$Q' = P(Q),$$
$$q_0' = \{q_0\},$$
$$F' = \{A \in P(Q) | A \cap F \neq \emptyset\},$$
$$\delta'(R, a) = \cup_{r \in R} \delta(r, a).$$

(b) Proof:

Lemma: $\delta^*(q, w) = \delta'^*(\{q\}, w)$.

Proof of Lemma: Do induction on $|w|$.

Base case: when $w = \epsilon$, $\delta^*(q, w) = \{q\} = \delta'^*(\{q\}, w)$.

Inductive case: assume lemma holds for $|w| \leq n, n \geq 0$. When $|w| = n + 1$, denote $w = xb, b \in \Sigma$. Then

$$\begin{aligned}
&\delta'^*(\{q\}, xb) \\
=&\delta'(\delta'^*(\{q\}, x), b) \\
=&\delta'(\delta^*(q, x), b) && \text{(by I.H.)} \\
=&\cup_{r \in \delta^*(q,x)} \delta(r, b) \\
=&\delta^*(q, xb)
\end{aligned}$$

i.e., lemma holds for $|w| = n + 1$.

By lemma, we have $L(M) = L(M')$ because

$$\begin{aligned}
&\forall w \in \Sigma^*, \\
&\qquad w \in L(M) \\
&\iff \delta^*(q_0, w) \cap F \neq \emptyset \\
&\iff \delta'^*(\{q_0\}, w) \cap F \neq \emptyset \\
&\iff \delta'^*(\{q_0\}, w) \in F' \\
&\iff w \in L(M')
\end{aligned}$$

Cont.

3. Show that for an NFA w/o epsilons, the two definitions of δ^* are equivalent, i.e., if we use the standard definition ($w = xa$), you want to show:

$$\delta^*(q, ax) = \cup_{p \in \delta(q,a)} \delta^*(p, x)$$

solution:

Prove by induction on $|w|$.

Base Case: when $w = a, a \in \Sigma$,

$$\begin{aligned}
&\delta^*(q, a) \\
&= \cup_{p \in \delta^*(q,\epsilon)} \delta(p, a) \\
&= \delta(q, a) \\
&= \cup_{p \in \delta(q,a)} p \\
&= \cup_{p \in \delta(q,a)} \delta^*(p, \epsilon)
\end{aligned}$$

Inductive Case: assume theorem holds for $|w| \leq n, n \geq 1$.

When $|w| = n + 1$, denote $|w| = axb, a, b \in \Sigma$.

$$\begin{aligned}
&\delta^*(q, axb) \\
&= \cup_{p \in \delta^*(q,ax)} \delta(p, b) \\
&= \cup_{p \in \cup_{p' \in \delta(q,a)} \delta^*(p',x)} \delta(p, b) && \text{(by I.H.)} \\
&= \cup_{p' \in \delta(q,a)} (\cup_{p \in \delta^*(p',x)} \delta(p, b)) \\
&= \cup_{p' \in \delta(q,a)} \delta^*(p, xb)
\end{aligned}$$

i.e., theorem holds for $|w| = n + 1$.

Cont.

4. Write at least three definitions of epsilon-closure. Also define the epsilon-closure of a set of states.

   solution:

   (also see slides "week4", page 2)

   def 0: $E(q) = \{p | p$ is reachable from q by 0 or more ϵ edges $\}$.

   def 1: $E(q)$ is the smallest set s.t.

   - $q \in E(q)$
   - if $q \in E(q)$, then $\delta(p, \epsilon) \subseteq E(q)$.

   def 2: $E(q) = \cup_i E_i(q)$

   - $E_0(q) = \{q\}$
   - $E_{i+1}(q) = \cup_{p \in E_i(q)} \delta(p, \epsilon)$

   closure of a set of states: $E(R) = \cup_{q \in R} E(q)$.

5. (Redo 2 for epsilons) For any given NFA with epsilons $M = (Q, \Sigma, \delta, q_0, F)$,

   (a) construct a DFA M' such that L(M) = L(M').

   (b) prove that $L(M) = L(M')$. i.e., $\forall w \in L(M)$, you want to show $w \in L(M')$, and $\forall w \in L(M')$, you want to show $w \in L(M)$.

solution:

(a)

$$M' = (Q', \Sigma, \delta', q_0', F'),$$

where

$$Q' = P(Q),$$

$$q_0' = \{q_0\},$$

$$F' = \{A \in P(Q) | A \cap F \neq \emptyset\},$$

$$\delta'(R, a) = E(\cup_{r \in R} \delta(r, a)), a \in \Sigma.$$

(b) Proof:

Lemma: $\delta^*(q, w) = \delta'^*(\{q\}, w)$.

Proof of Lemma: Do induction on $|w|$.

Base case: when $w = \epsilon$, $\delta^*(q, \epsilon) = E(\{q\})$, $\delta'^*(\{q\}, \epsilon) = E(\cup_{r \in \{q\}} \delta(r, \epsilon)) = E(\delta(q, \epsilon)) = E(\{q\})$, i.e. lemma holds.

Inductive case: assume lemma holds for $|w| \leq n, n \geq 0$. When $|w| = n + 1$, denote $w = xb, b \in \Sigma$. Then

$$\begin{aligned} & \delta'^*(\{q\}, xb) \\ =& \delta'(\delta'^*(\{q\}, x), b) \\ =& \delta'(\delta^*(q, x), b) && \text{(by I.H.)} \\ =& E(\cup_{r \in \delta^*(q,x)} \delta(r, b)) \\ =& \delta^*(q, xb) \end{aligned}$$

i.e., lemma holds for $|w| = n + 1$.

By lemma, we have $L(M) = L(M')$ because

$$\begin{aligned} \forall w \in \Sigma^*, & \\ & w \in L(M) \\ \iff & \delta^*(q_0, w) \cap F \neq \emptyset \\ \iff & \delta'^*(\{q_0\}, w) \cap F \neq \emptyset \\ \iff & \delta'^*(\{q_0\}, w) \in F' \\ \iff & w \in L(M') \end{aligned}$$

(Hint 1: this definition of δ' is equivalent as the one on textbook (since $E(\cup_i S_i) = \cup_i (E(S_i))$.))

(Hint 2: From the definition of E, we can prove $E(\cup_i S_i) = \cup_i (E(S_i))$ and $E(E(S)) = E(S)$.)

6. (Redo 3 for epsilons) Figure out an alternative definition of δ^* for NFA with epsilons, and prove the equivalence.

solution:

Alternative definition of δ^* on $|w| > 0$:

$$\delta^*(q, w) = \cup_{p \in E(\delta(q,a))} \delta^*(p, x), w = ax, a \in \Sigma.$$

Now prove for the original definition of δ^*, and $|w| > 0$, we have

$$\delta^*(q, w) = \cup_{p \in E(\delta(q,a))} \delta^*(p, x), w = ax, a \in \Sigma.$$

Prove by induction on $|w|$.

Base Case: when $w = a, a \in \Sigma$,

$$\begin{aligned}
&\delta^*(q, a) \\
=&E(\cup_{p \in \delta^*(q,\epsilon)} \delta(p, a)) \\
=&E(\delta(q, a))
\end{aligned}$$

$$\begin{aligned}
&\cup_{p \in E(\delta(q,a))} \delta^*(p, \epsilon) && \\
=& \cup_{p \in E(\delta(q,a))} E(\{p\}) && \\
=&E(\cup_{p \in E(\delta(q,a))} \{p\}) && (E\cup = \cup E) \\
=&E(E(\delta(q, a))) && \\
=&E(\delta(q, a)) && (E \cdot E = E)
\end{aligned}$$

i.e., theorem holds $w = a$.

Inductive Case: assume theorem holds for $|w| \leq n, n \geq 1$.

When $|w| = n + 1$, denote $|w| = axb, a, b \in \Sigma$.

$$\begin{aligned}
&\delta^*(q, axb) && \\
=&E(\cup_{p \in \delta^*(q,ax)} \delta(p, b)) && \\
=&E(\cup_{p \in (\cup_{p' \in E(\delta(q,a))} \delta^*(q',x))} \delta(p, b)) && \text{(by I.H.)} \\
=&E(\cup_{p' \in E(\delta(q,a))} \cup_{p \in \delta^*(p',x)} \delta(p, b)) && \\
=& \cup_{p' \in E(\delta(q,a))} E(\cup_{p \in \delta^*(p',x)} \delta(p, b)) && (E\cup = \cup E) \\
=& \cup_{p' \in E(\delta(q,a))} \delta^*(p, xb) &&
\end{aligned}$$

i.e., theorem holds for $|w| = n + 1$.

7. Devise an algorithm to convert an NFA with epsilons to an epsilon-free NFA.

   solution:

Algorithm 1 Algorithm for converting an NFA to an ϵ-free NFA

Input: an NFA $M = \{Q, \Sigma, \delta, q_0, F\}$
Outout: an NFA $M' = \{Q', \Sigma, \delta', q'_0, F'\}$, $\delta' : Q \times \Sigma \mapsto P(Q)$

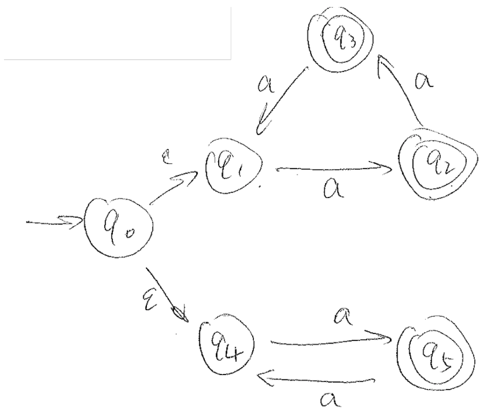
1: **repeat**
2: $\quad flag := False$
3: $\quad$ **for** $q \in Q$ **do**
4: $\quad\quad$ **if** $flag = True$ **then**
5: $\quad\quad\quad$ break
6: $\quad\quad$ **end if**
7: $\quad\quad S := E(q)$
8: $\quad\quad$ **if** $S \neq \{q\}$ **then**
9: $\quad\quad\quad$ **for** $a \in \Sigma$ **do**
10: $\quad\quad\quad\quad \delta(q, a) := \cup_{p \in S} \delta(p, a)$
11: $\quad\quad\quad$ **end for**
12: $\quad\quad\quad \delta(q, \epsilon) := \delta(q, \epsilon) - S$
13: $\quad\quad\quad$ **if** $S \cap F \neq \emptyset$ **then**
14: $\quad\quad\quad\quad F := F \cup \{q\}$
15: $\quad\quad\quad$ **end if**
16: $\quad\quad\quad flag := True$
17: $\quad\quad$ **end if**
18: $\quad$ **end for**
19: **until** $flag = False$
20: **return** $(Q, \Sigma, \delta, q_0, F)$

(Hint: a faster version would be processing all the states in S in one loop, instead of processing q only. (from line 8 to line 17))

8. Convert the NFAs from the last two questions in Quiz 3 to DFAs. Quiz 3 solutions are on canvas.

solution:

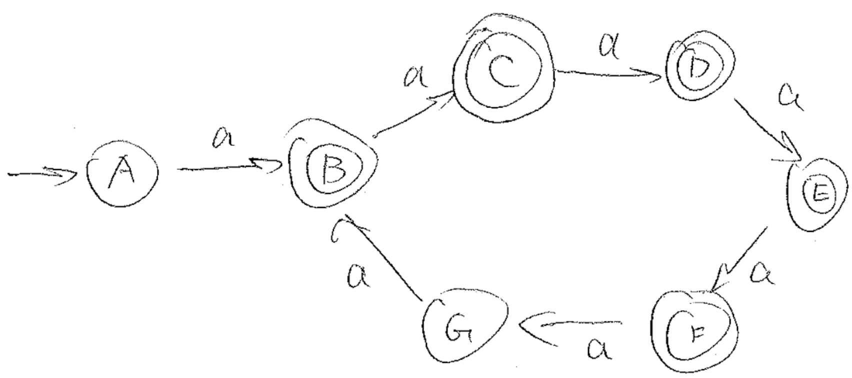
(a) NFA:



conversion:

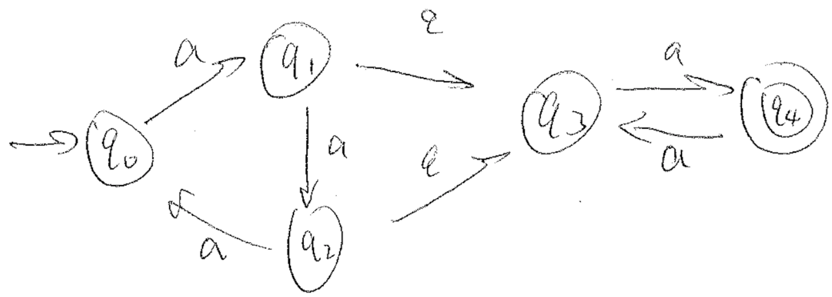
	a
$A = \{q_0, q_1, q_4\}$	B
$B = \{q_2, q_5\}$	C
$C = \{q_3, q_4\}$	D
$D = \{q_1, q_5\}$	E
$E = \{q_2, q_4\}$	F
$F = \{q_3, q_5\}$	G
$G = \{q_1, q_4\}$	C

DFA:



(Hint: states A and G could be merged here.)

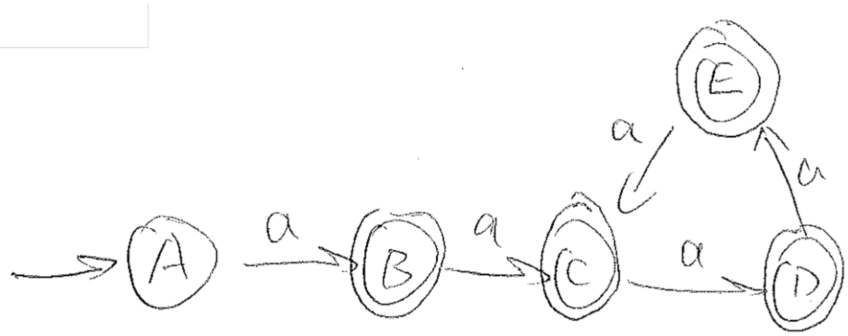
(b) NFA:



conversion:

	a
$A = \{q_0\}$	B
$B = \{q_1, q_3\}$	C
$C = \{q_2, q_3, q_4\}$	D
$D = \{q_0, q_3, q_4\}$	E
$E = \{q_1, q_3, q_4\}$	C

DFA:



(Hint: states B, C, D and E could be merged here.

9. Why it is important to add “the smallest set” in Def. 1 of epsilon-closure? Give an example where dropping “the smallest” doesn’t make sense.

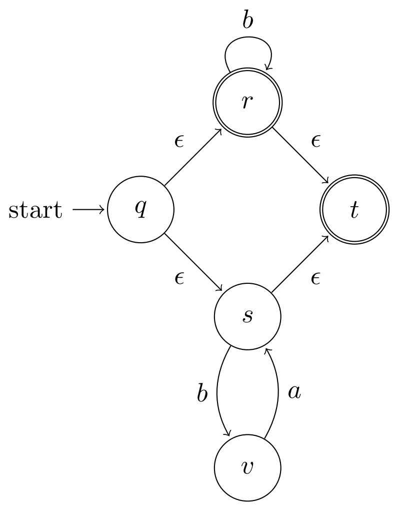
   solution:

   “the smallest set” guarantees that all states in $E(q)$ would be reachable from q by only using zero to many numbers of ϵ.

   Consider NFA in question 8 (b). If “the smallest” is dropped, q_1, q_3, q_4 satisfies the definition of $E(q_1)$. However, q_1 cannot reach q_4 by only using any number of ϵ.

Cont.

10. For each of the following NFAs, do
    (a) compute epsilon-closure for each state,
    (b) convert it to a DFA,
    (c) explain the intuitive meaning of the language
    (d) try to find a smaller but equivalent DFA, if any.



Now flip the two links between s and v and redo everything. (i.e., from s on a goes to v, from v on b goes back to s).

Now further add an ϵ link from t to r and redo everything.

solution:

(a1)

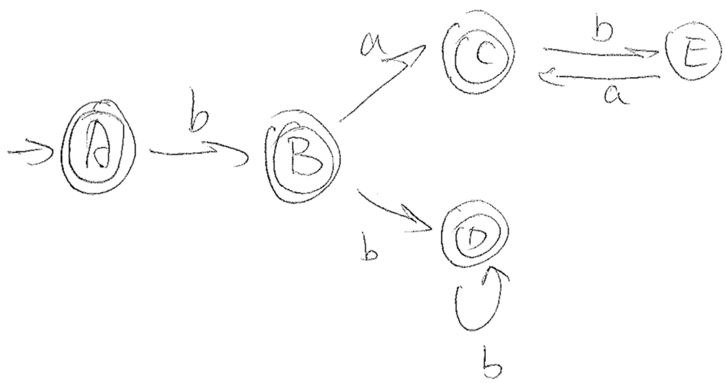
x	$E(x)$
q	$\{q, r, s, t\}$
r	$\{r, t\}$
s	$\{s, t\}$
t	$\{t\}$
v	$\{v\}$

(b1)

conversion:

	a	b
$A = \{q, r, s, t\}$		B
$B = \{r, t, v\}$	C	D
$C = \{s, t\}$		E
$D = \{r, t\}$		D
$E = \{v\}$	C	

DFA:



(c1)

This languages contains strings with the format b^* or strings with the format $(ba)^*$, but not other strings.

(d1)

DFA in (b1) is the smallest.

(a2)

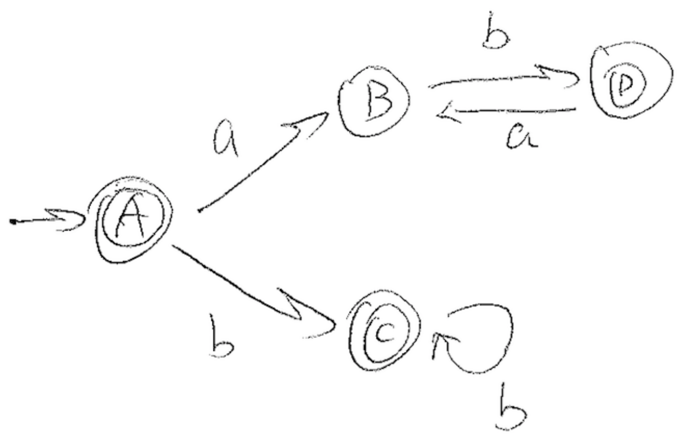
same as (a1).

(b2)

conversion:

	a	b
$A = \{q, r, s, t\}$	B	C
$B = \{v\}$		D
$C = \{r, t\}$		C
$D = \{s, t\}$	B	

DFA:



(c2)

This languages contains strings with the format b^* or strings with the format $(ab)^*$, but not other strings.

(d2)

DFA in (b2) is the smallest.

(a3)

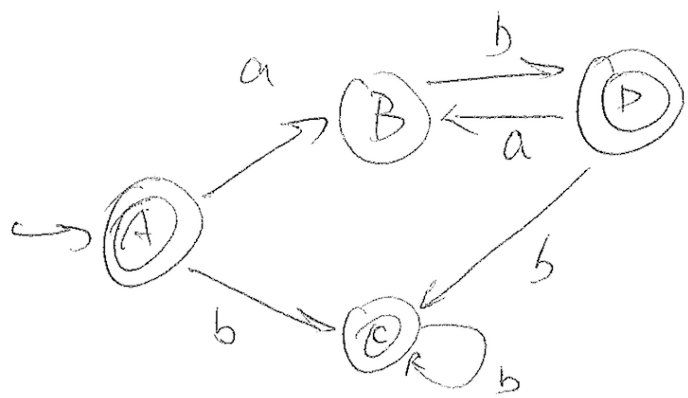
x	$E(x)$
q	$\{q, r, s, t\}$
r	$\{r, t\}$
s	$\{r, s, t\}$
t	$\{r, t\}$
v	$\{v\}$

(b3)

conversion:

	a	b
$A = \{q, r, s, t\}$	B	C
$B = \{v\}$		D
$C = \{r, t\}$		C
$D = \{r, s, t\}$	B	C

DFA:



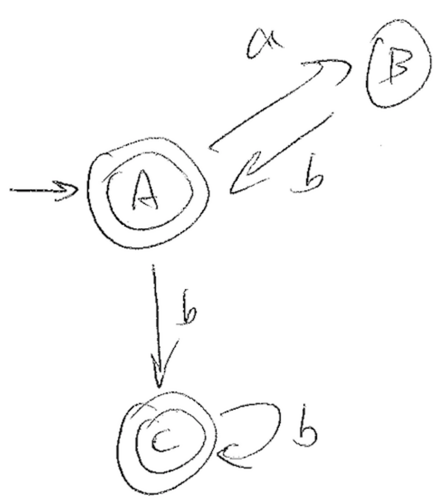
(c3)

This language contains strings with the format $(ab)^*b^*$, but not other strings.

(d3)

State A and D could be merged.

DFA:



The End.