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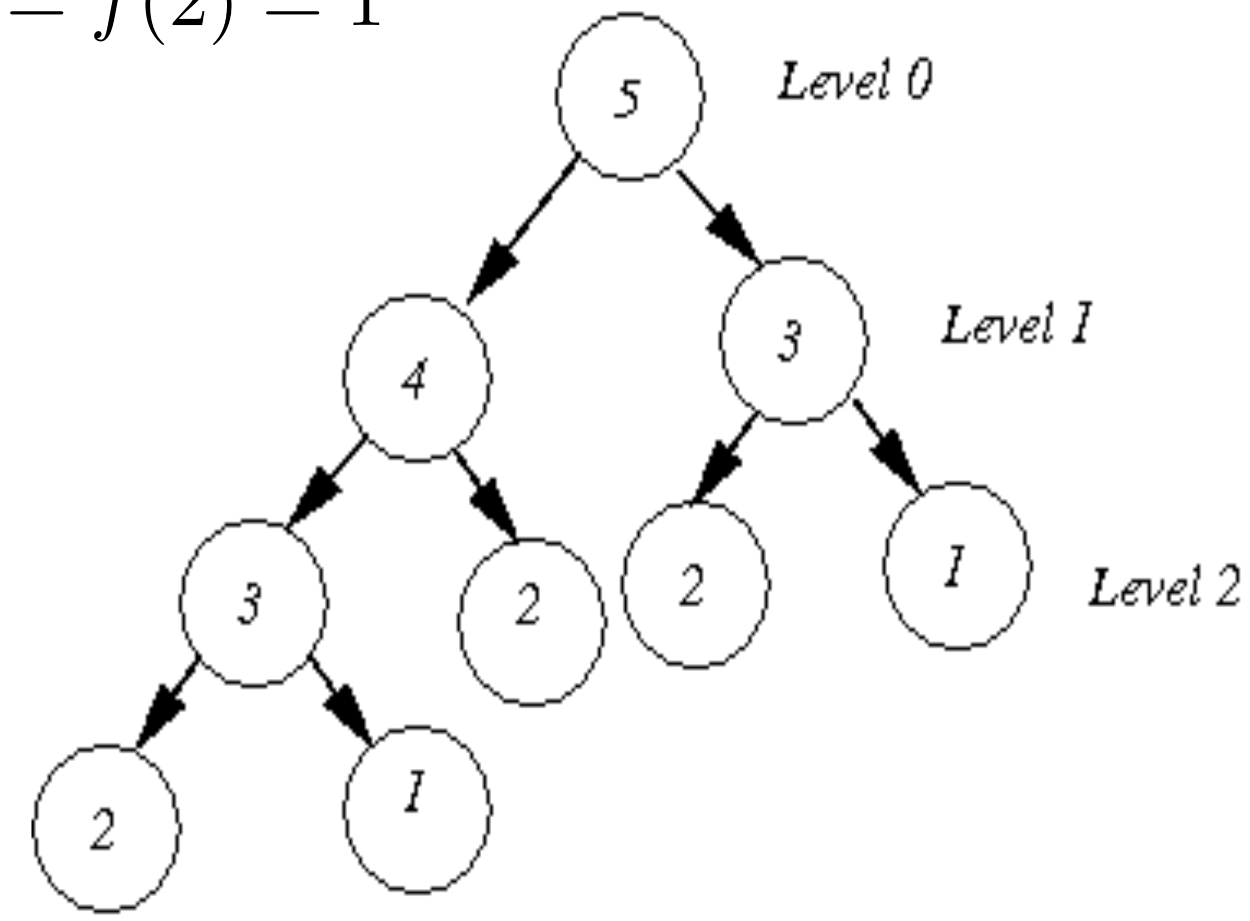

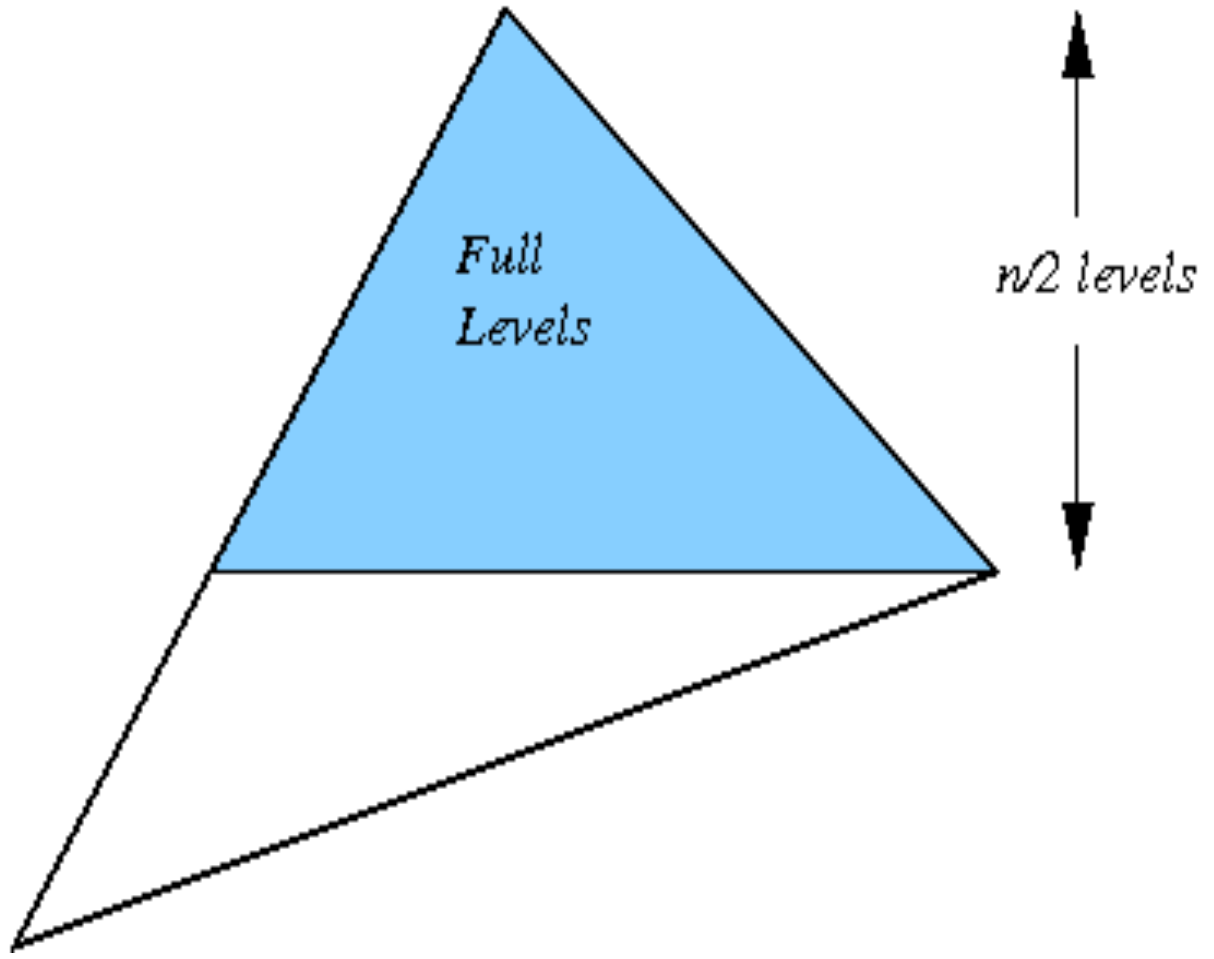

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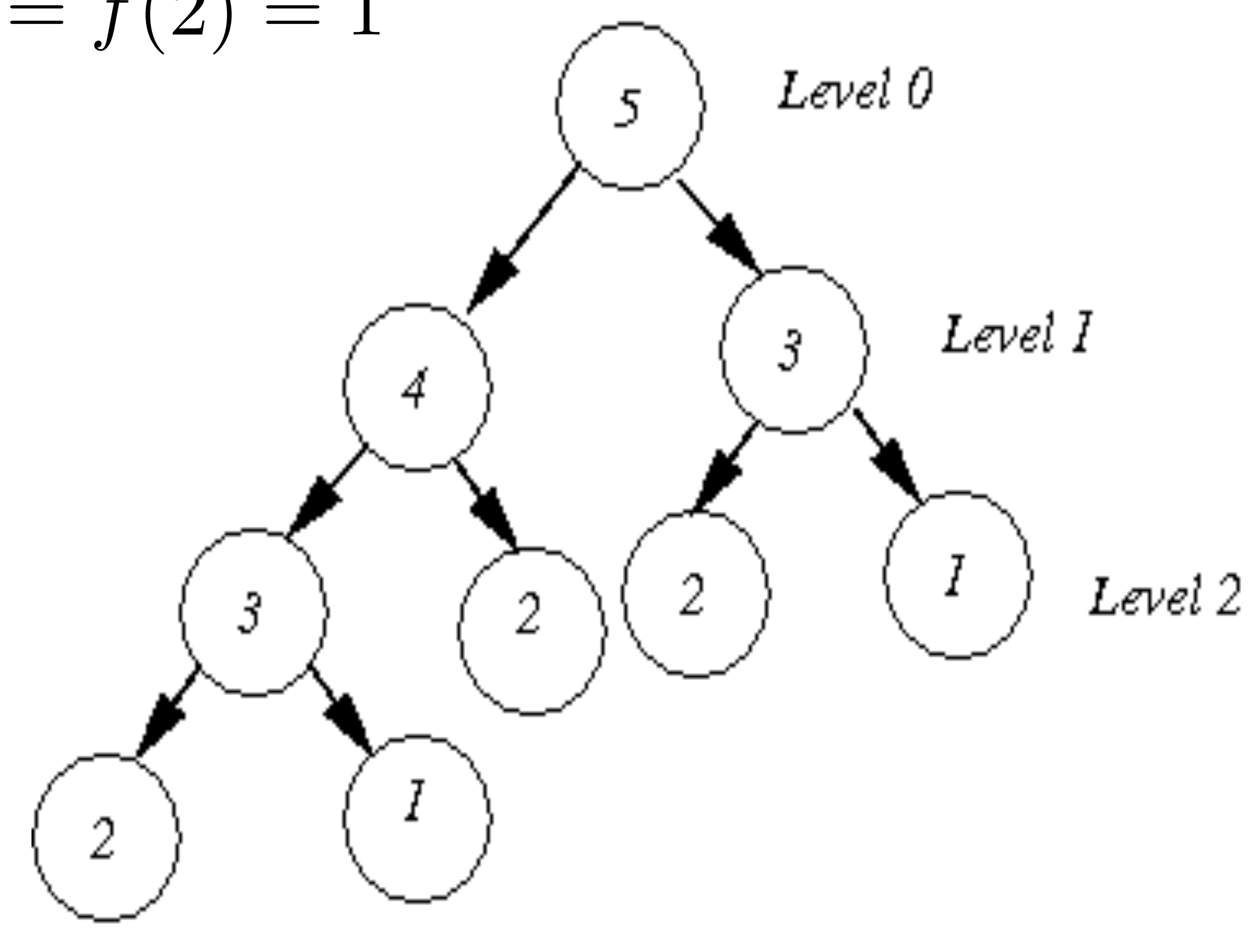

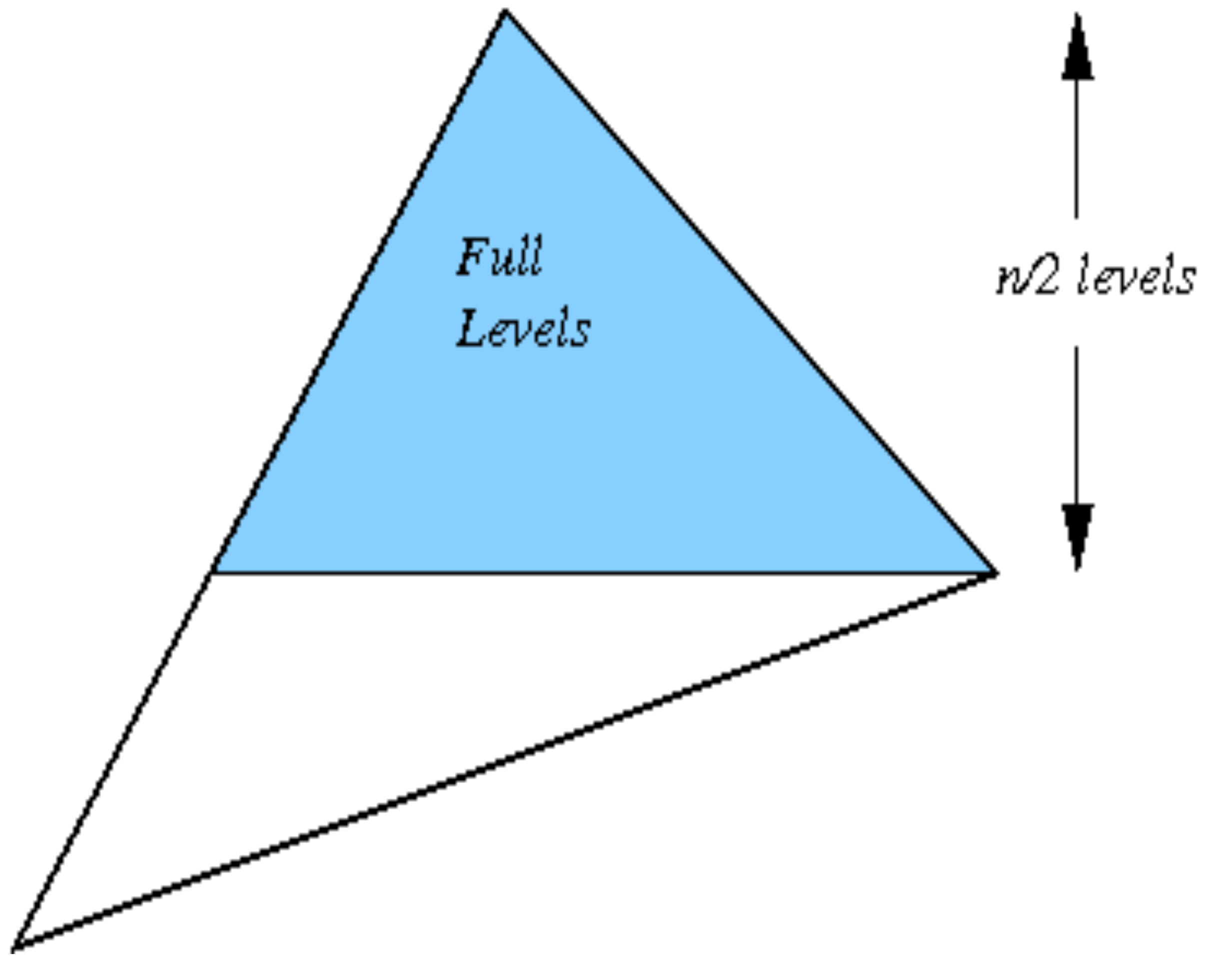

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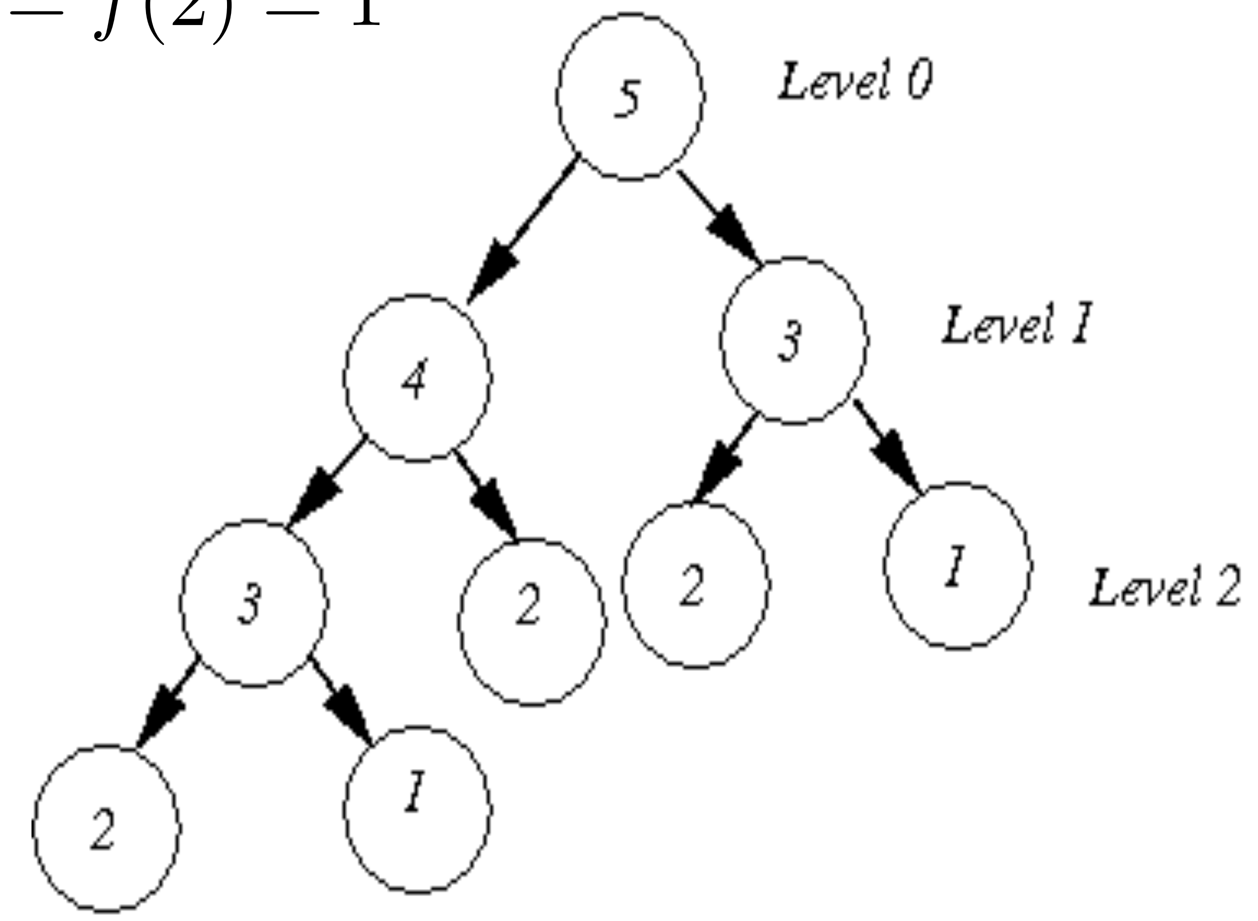

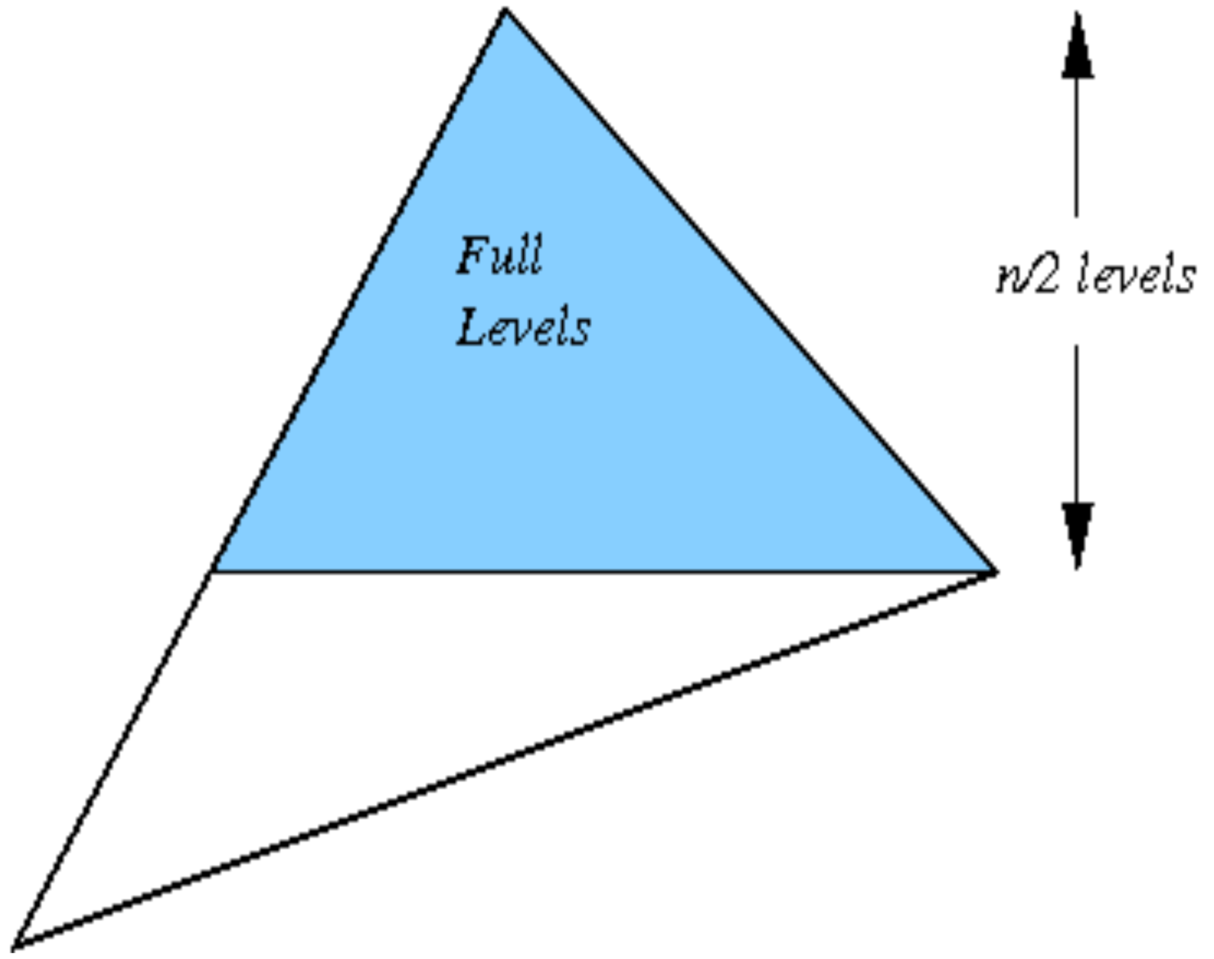

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DP1: top-down with memoization: $O(n)$

```
fibs={1:1, 2:1} # hash table (dict)
def fib1(n):
  if n not in fibs:
    fibs[n] = fib1(n-1) + fib1(n-2)
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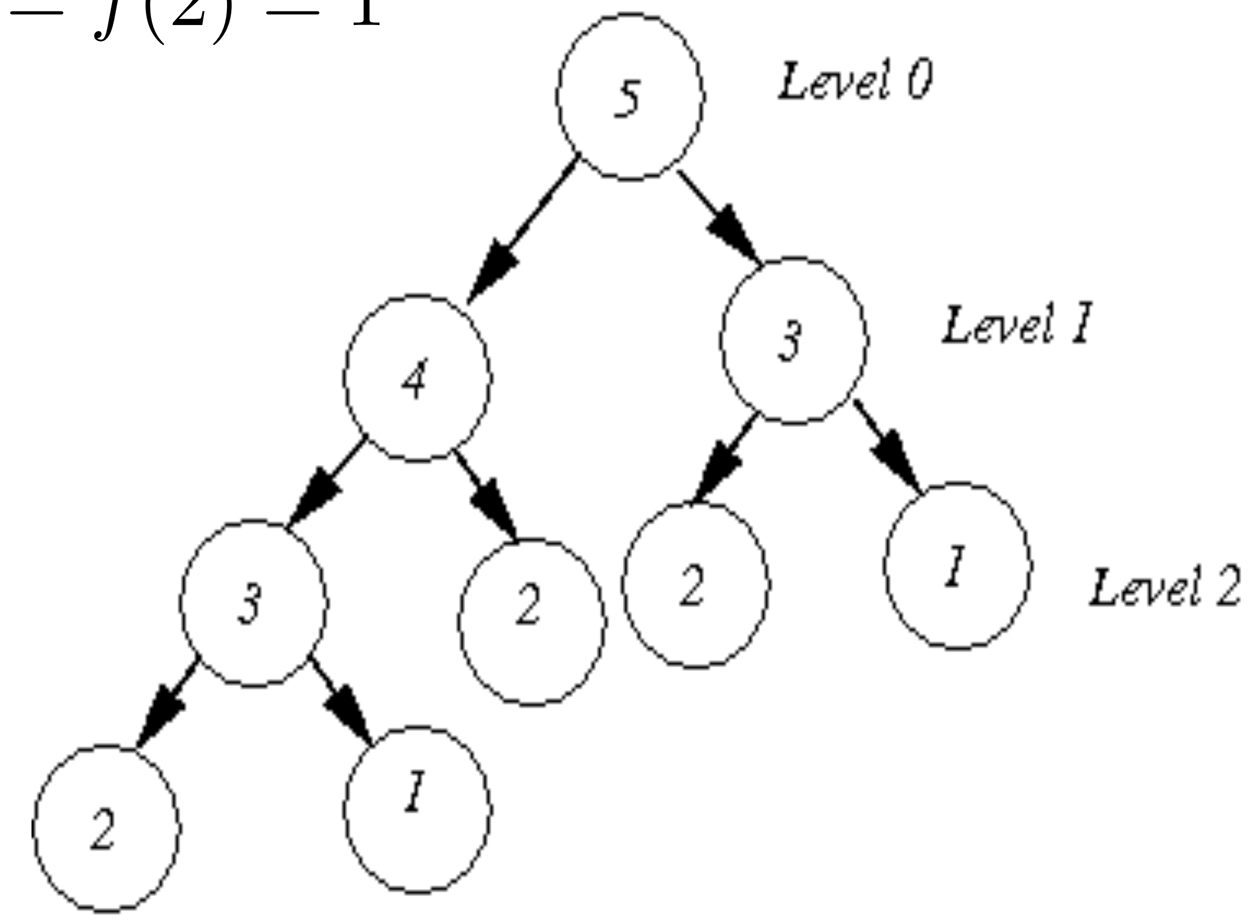

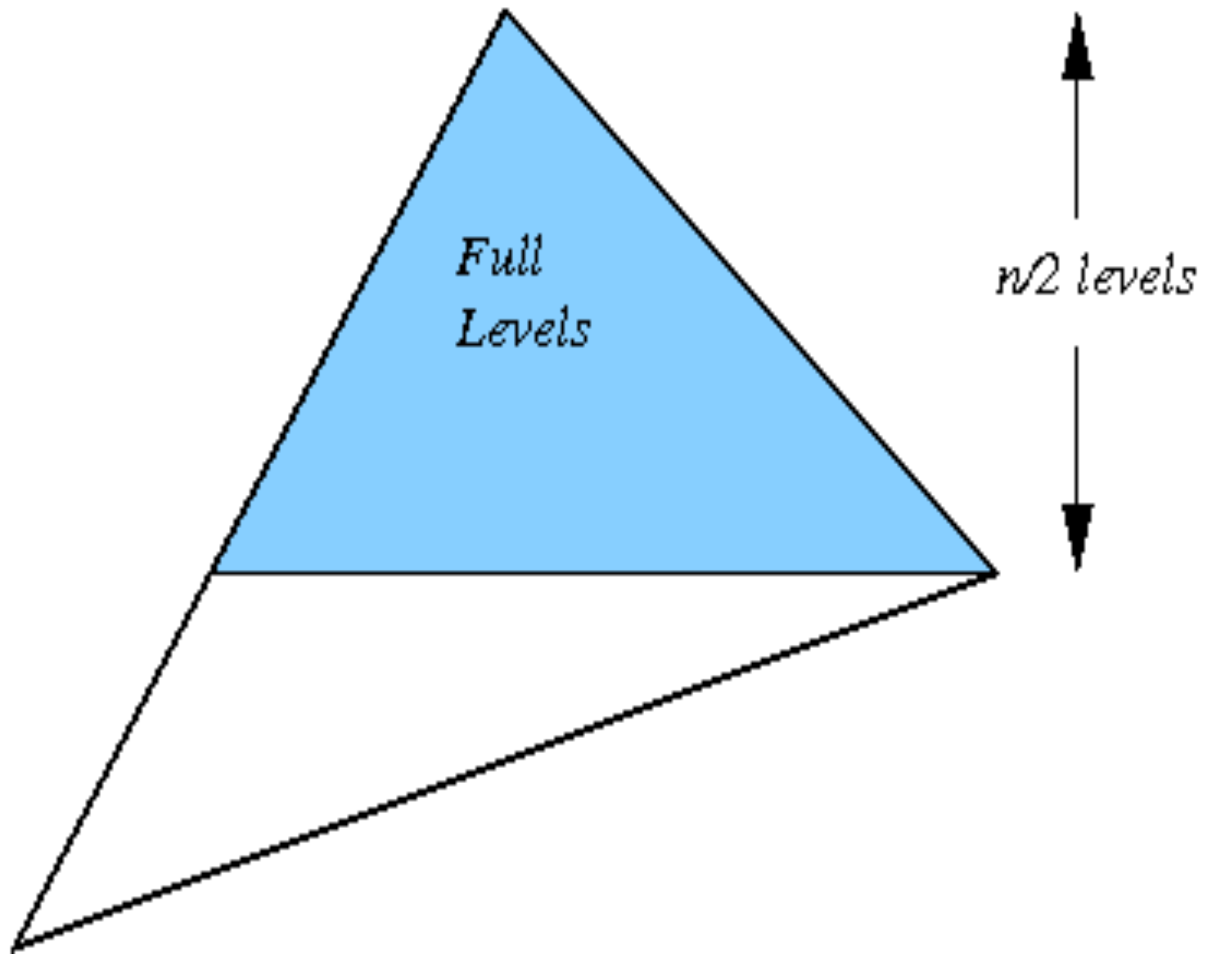

naive recursion without memoization: $O(1.618...^n)$

DP2: bottom-up: $O(n)$

```
def fib0(n):
  a, b = 1, 1
  for i in range(3, n+1):
    a, b = a+b, a
  return a
```

```
def fib0(n):
  f = [1, 1]
  for i in range(3, n+1):
    fibs.append(f[-1]+f[-2])
  return f[-1]
```

DP1: top-down with memoization: $O(n)$

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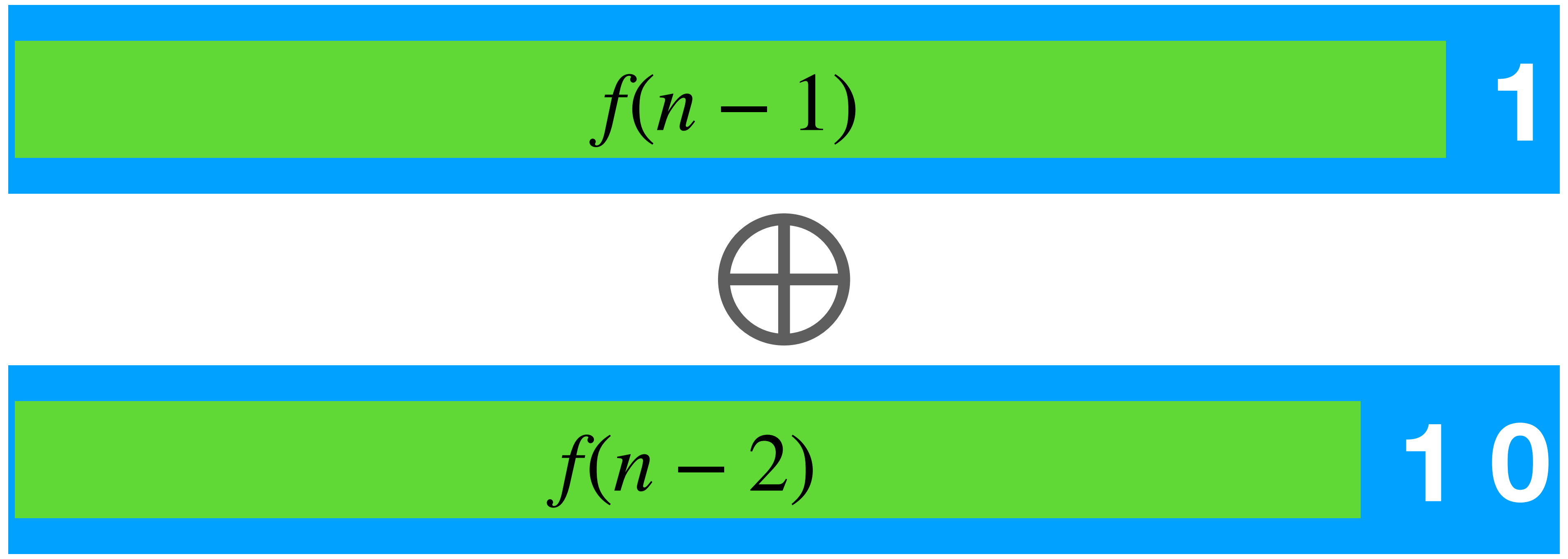

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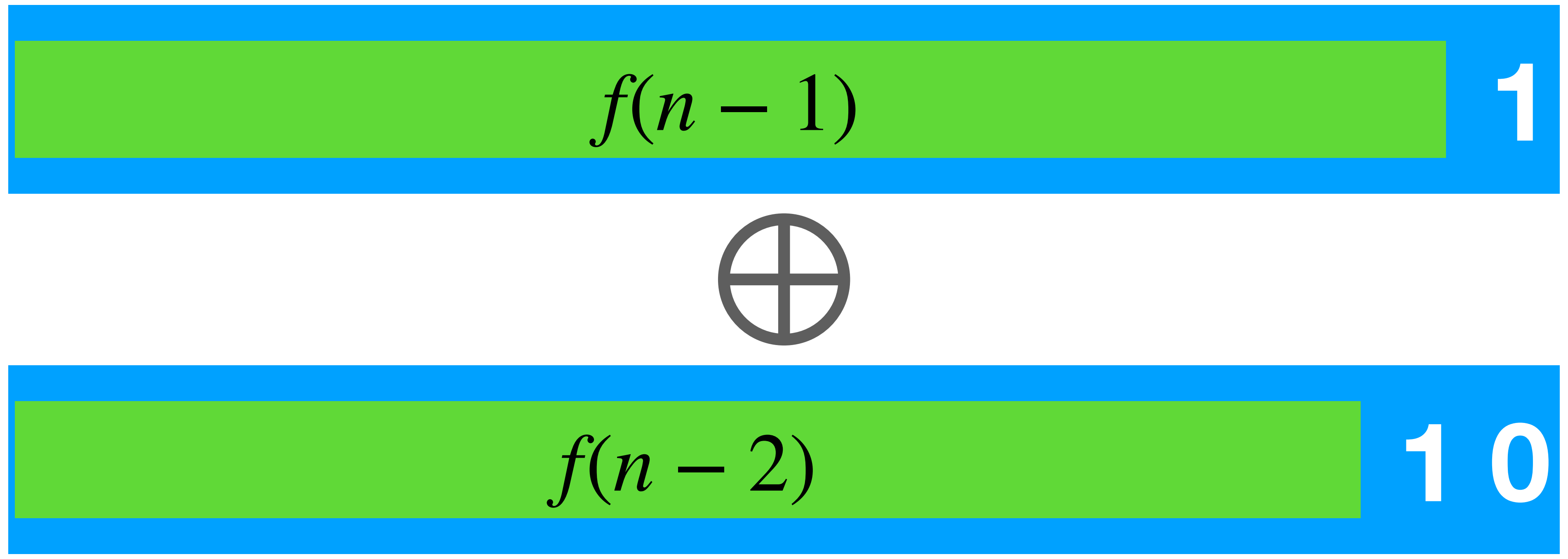

$$f(n) = f(n-1) + f(n-2)$$

$$f(1) = 2, \quad f(0) = 1$$

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- max weighted independent set on a linear-chain graph
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i	-1	0	1	2	3	4	5	6
a[i]			9	10	8	5	2	4
f(i)								

best value

backpointer

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									← *start here*

recursively backtrack
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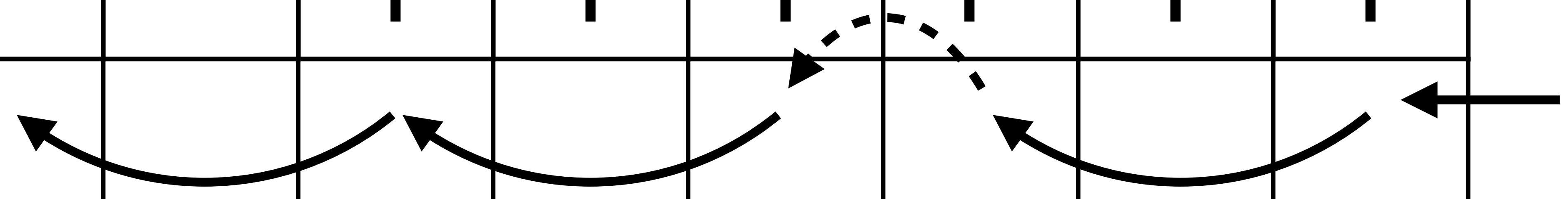

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  - subproblem: *f*(*i*) -- max independent set for a[1]..a[*i*] (1-based index)

$$f(i) = \max\{f(i-1),\ f(i-2) + a[i]\}$$

$$b(i) = [f(i) \neq f(i-1)]$$: take $a[i]$ for $f(i)$?

$f(0) = 0; f(1) = a[1]$?

No! $f(1) = \max\{a[1], 0\}$

or even better: $f(0) = 0;\ f(-1) = 0$

i	-1	0	1	2	3	4	5	6	
a[i]			9	10	8	5	2	4	
f(i)	0	0	9	10	17	17	19	21	*best value*
b(i)			T	T	T	F	T	T	*backpointer*
back track	*		* take		* take	* not		* take	← *start here*

recursively backtrack the optimal solution

Max Independent Set (MIS)

- max weighted independent set on a linear-chain graph
  - e.g. **9** — 10 — **8** — 5 — 2 — **4** ; best MIS: [9, 8, 4] = 21 (vs. greedy: [10, 5, 4] = 19)
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best value

backpointer

start here

recursively backtrack the optimal solution

MIS

$$f(n) = \max \begin{cases} f(n-1) + 0 \\ f(n-2) + a[n] \end{cases}$$

cost   reward

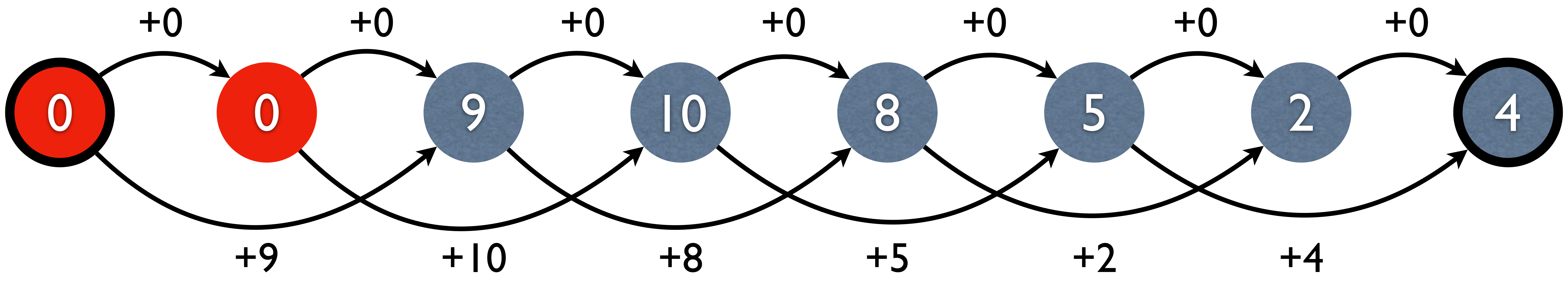
bitstrings

$$f(n) = + \begin{cases} f(n-1) \times 1 \\ f(n-2) \times 1 \end{cases}$$

summary operator ⊕ (across divides)

combination operator ⊗ (within a divide)

Graph Interpretation of DP



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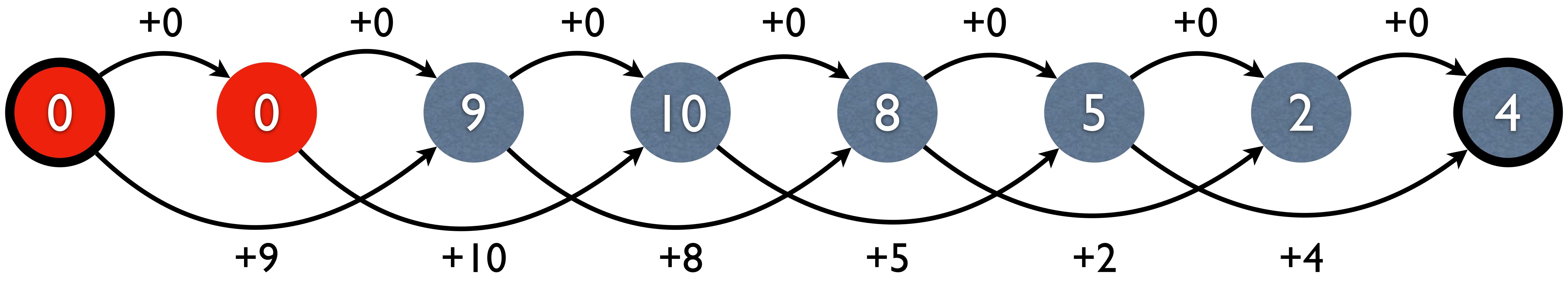
- MIS: longest path between source and target (see lecture video)
  - each node i has two incoming edges: $(i-2) \xrightarrow{a[i]} i$ (take) and $(i-1) \xrightarrow{0} i$ (not take)



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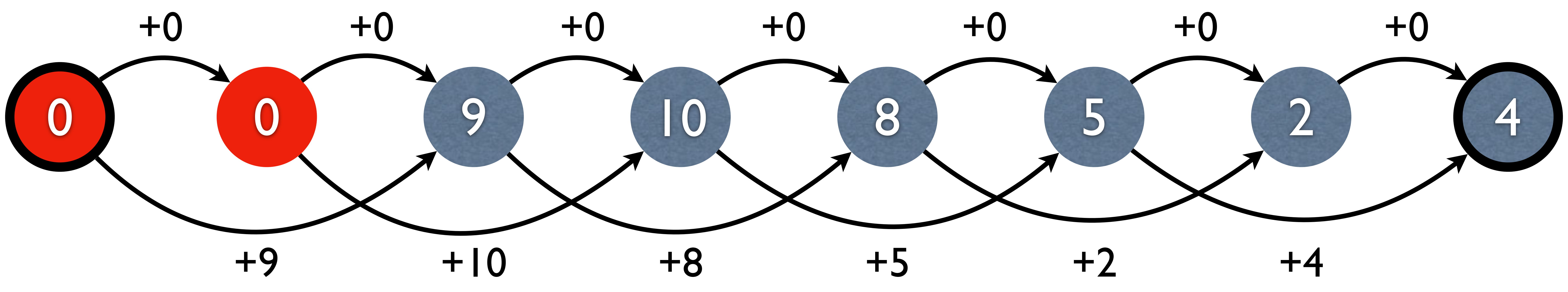

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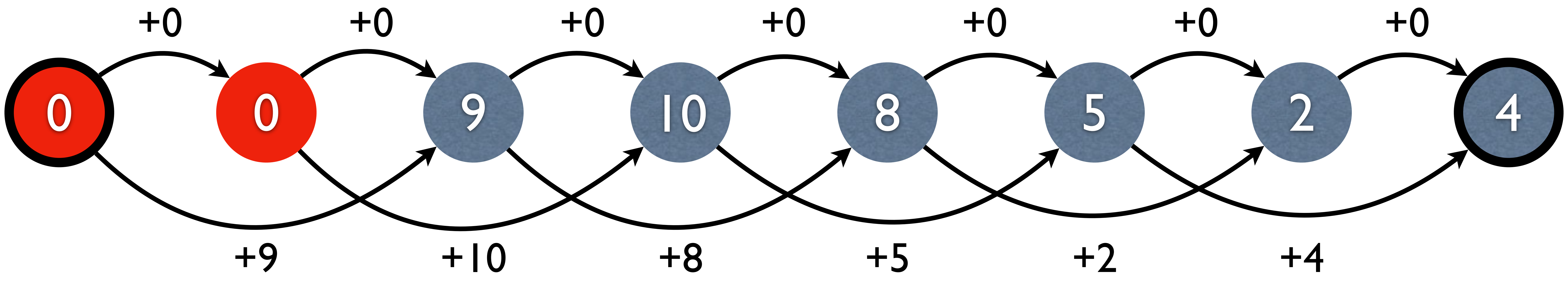

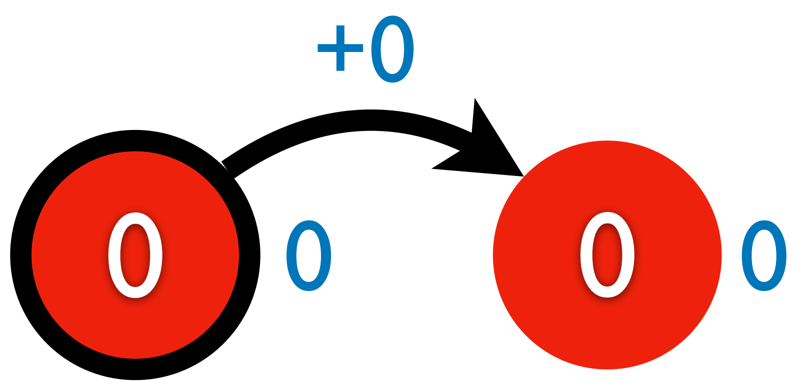

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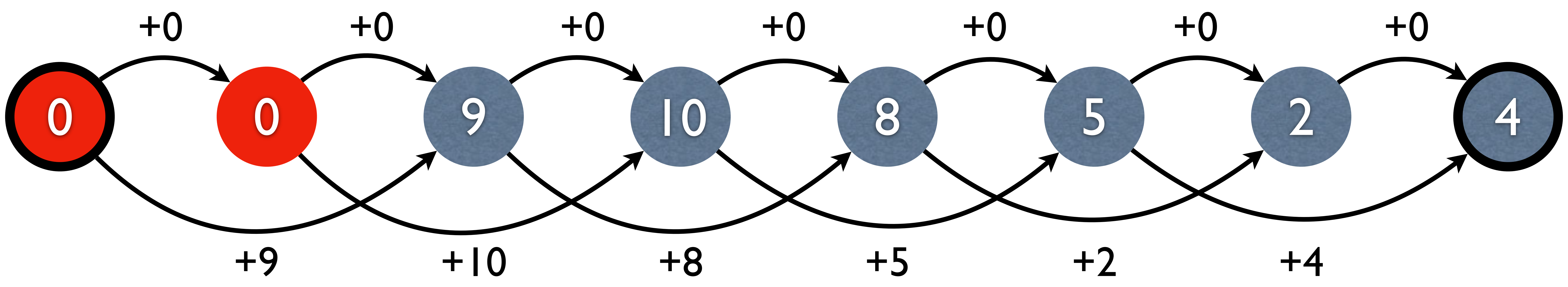

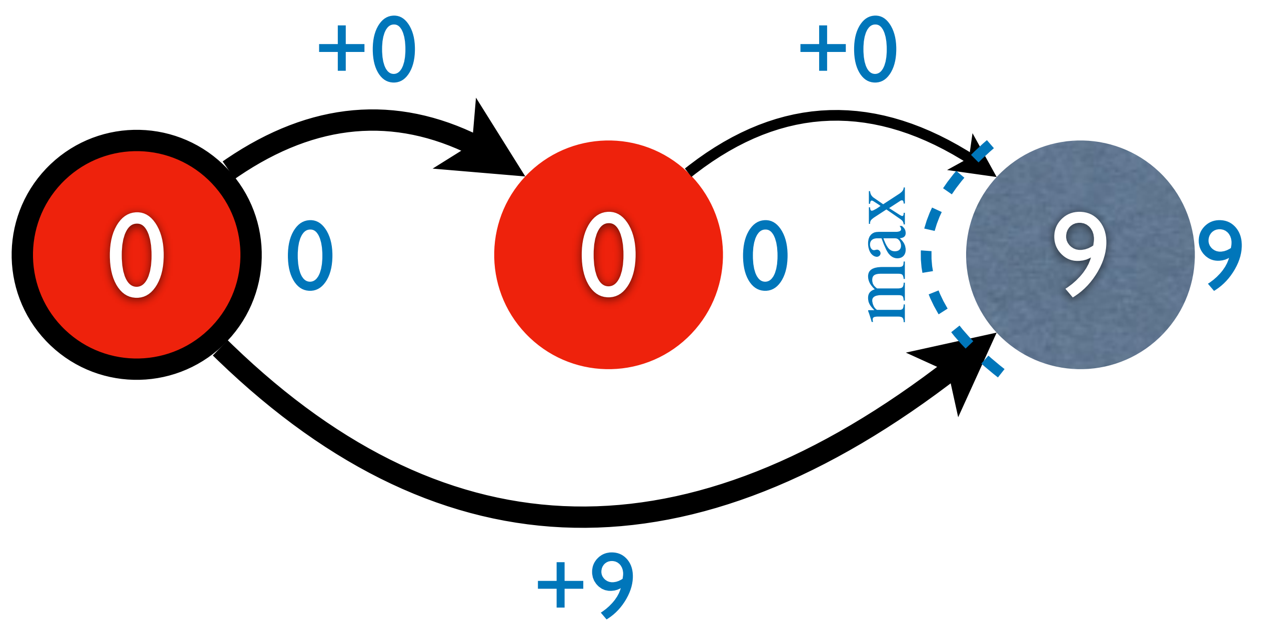

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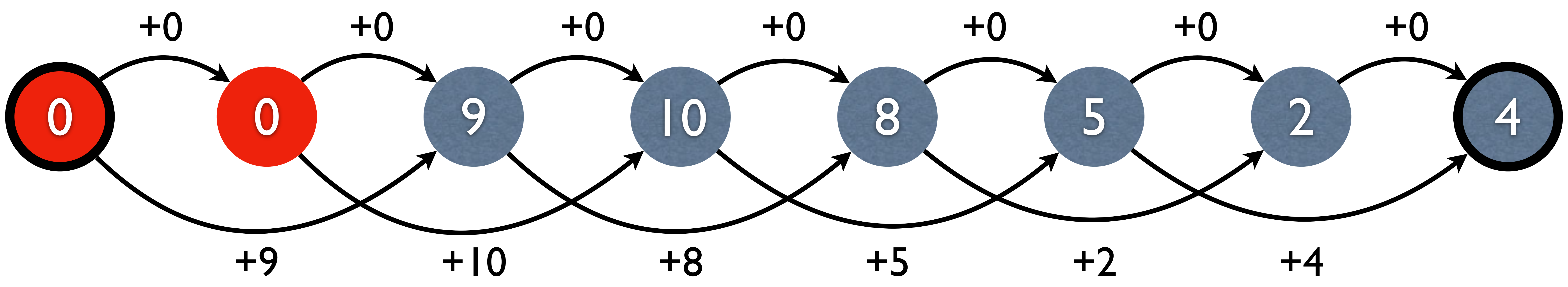

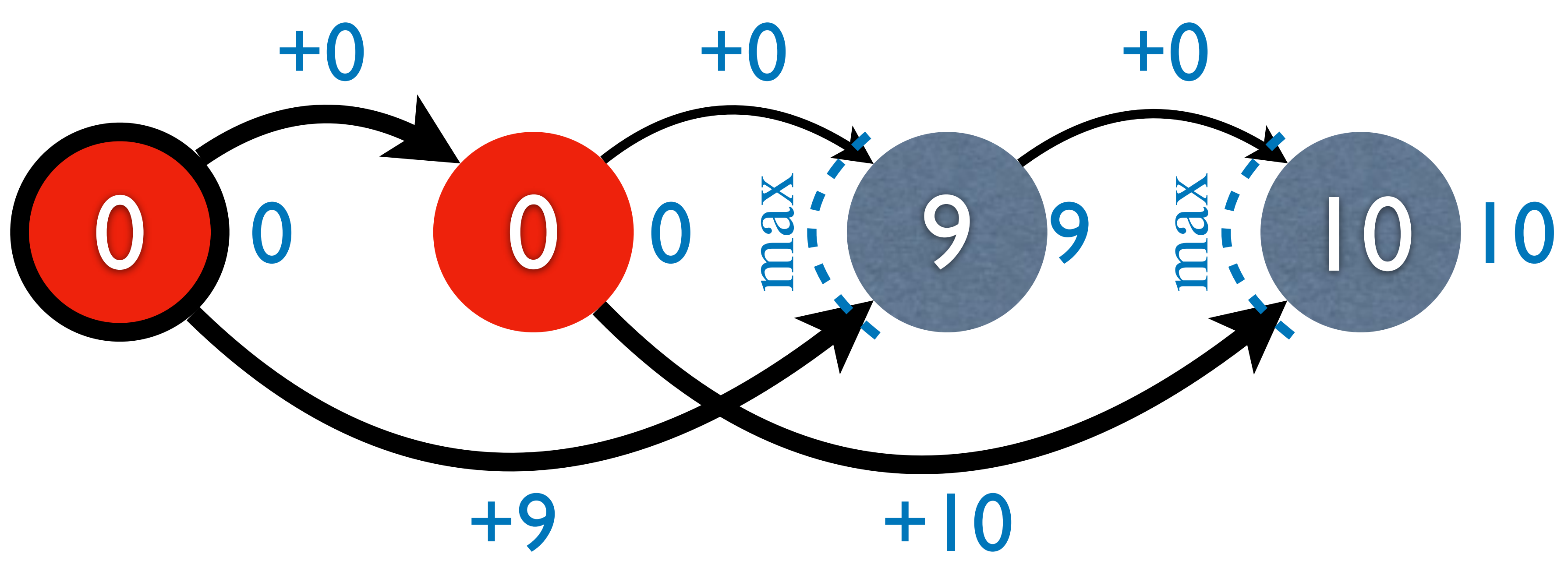

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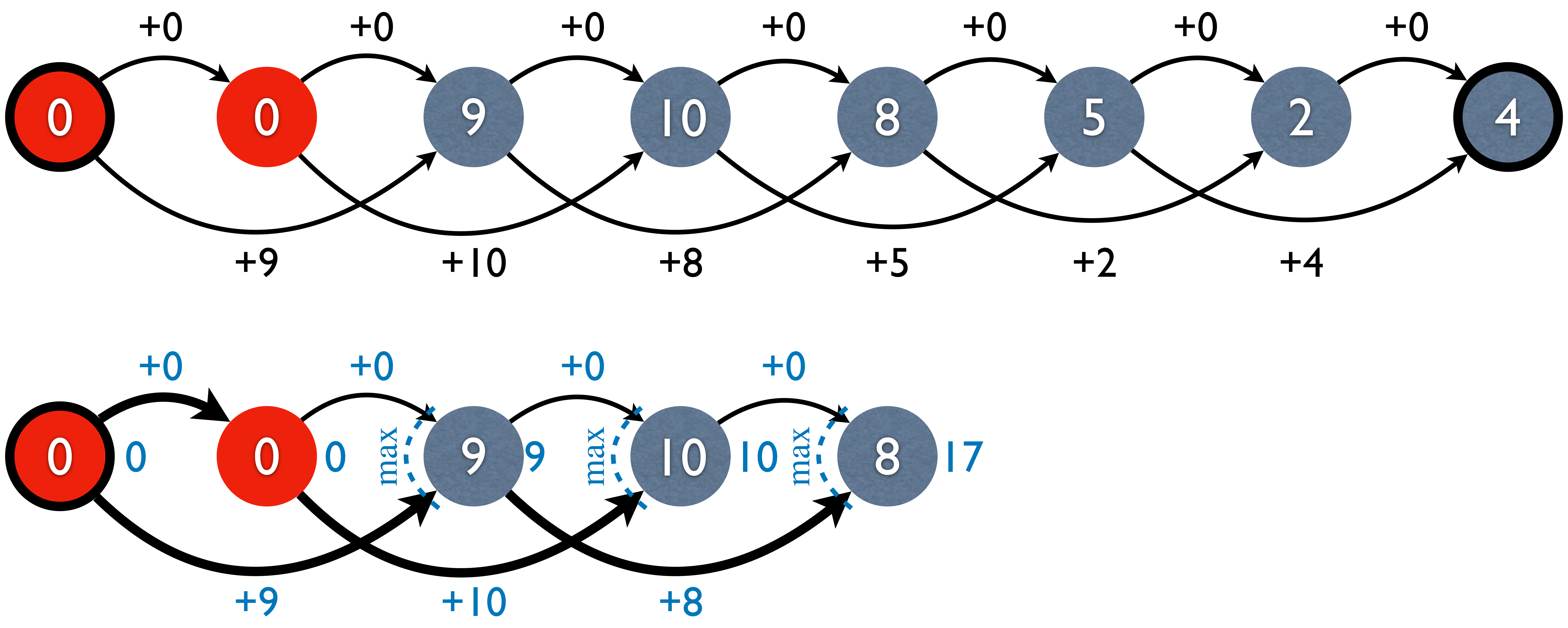

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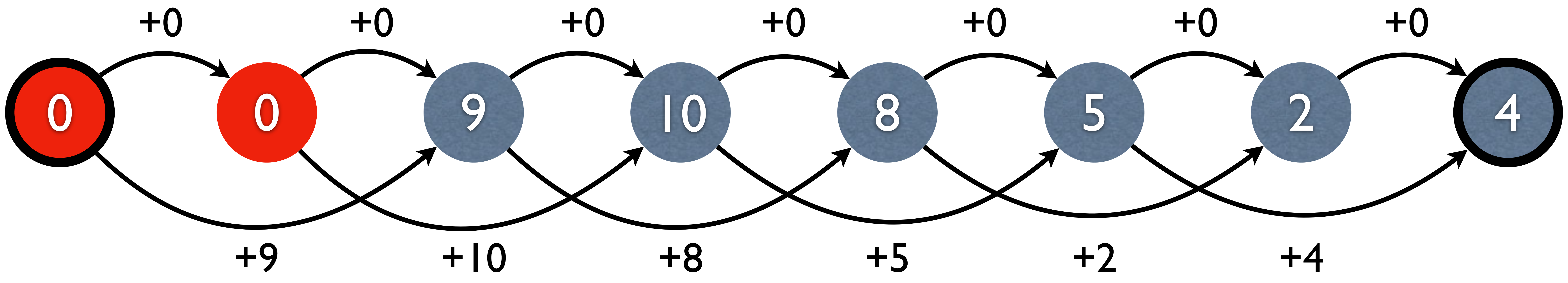

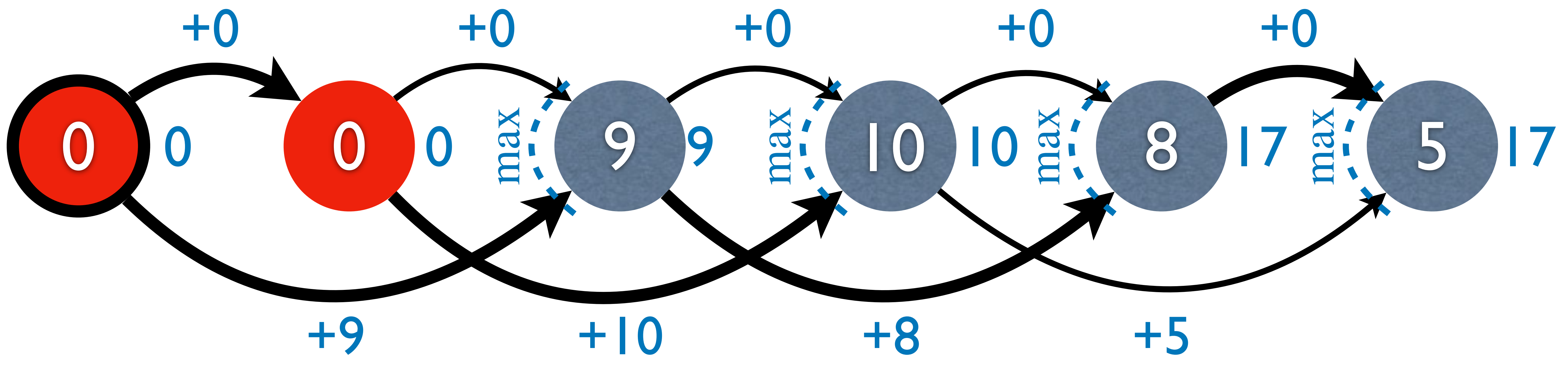

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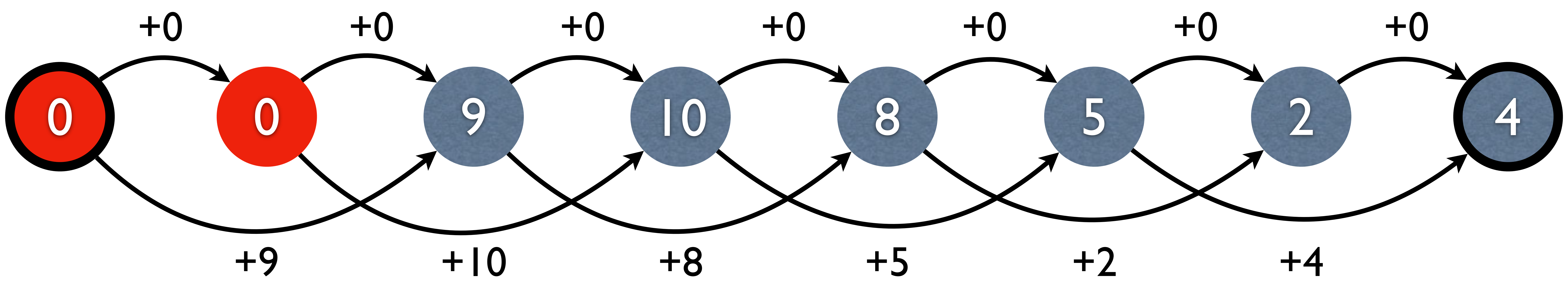

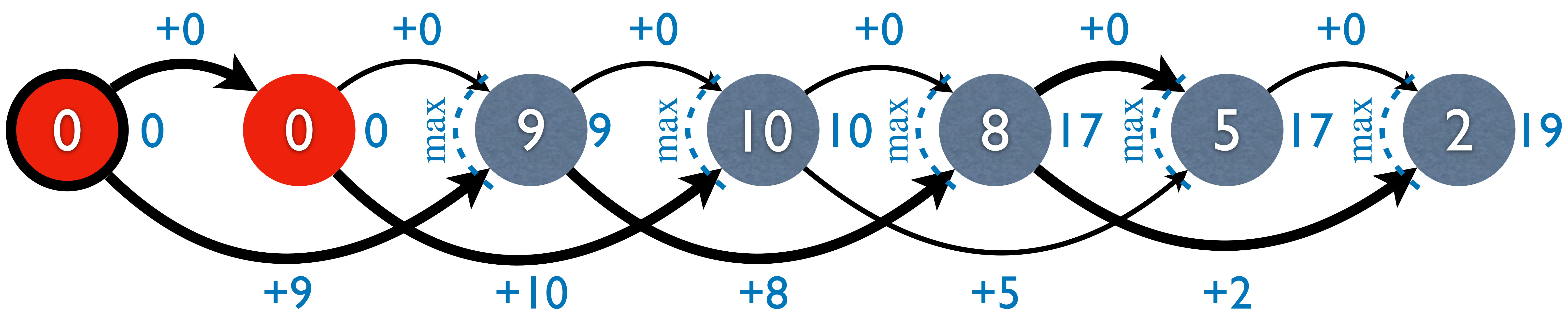

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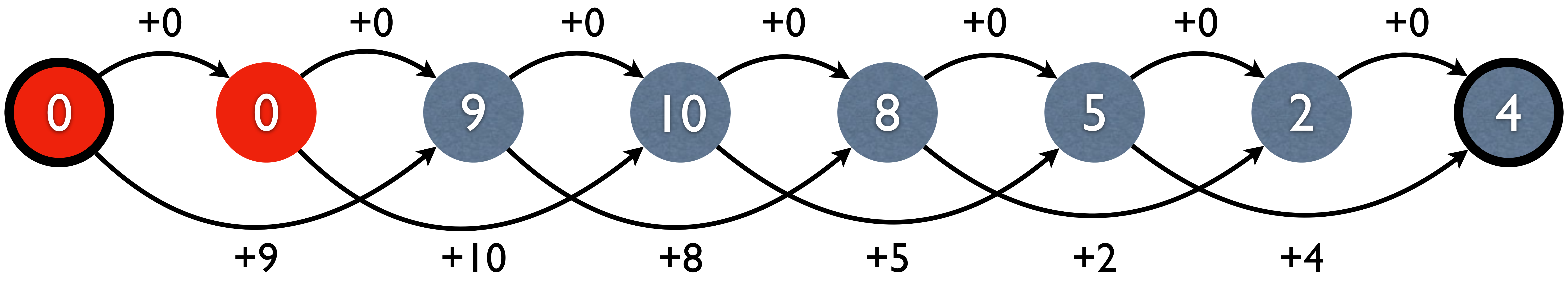

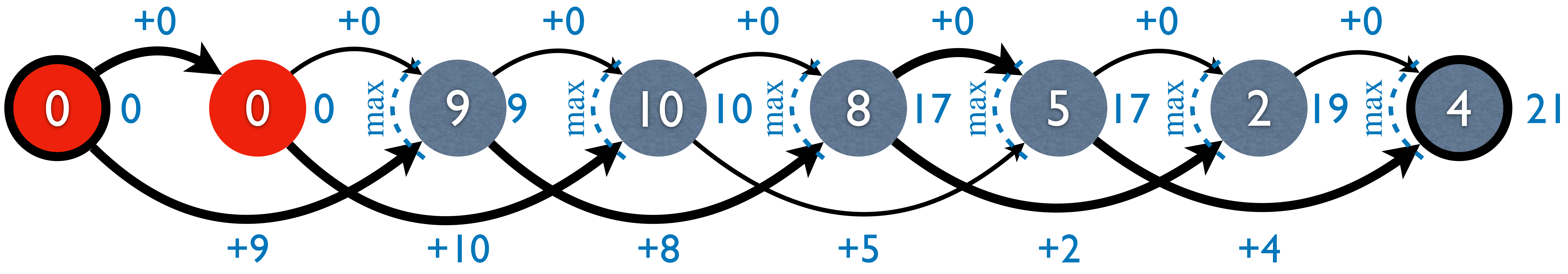

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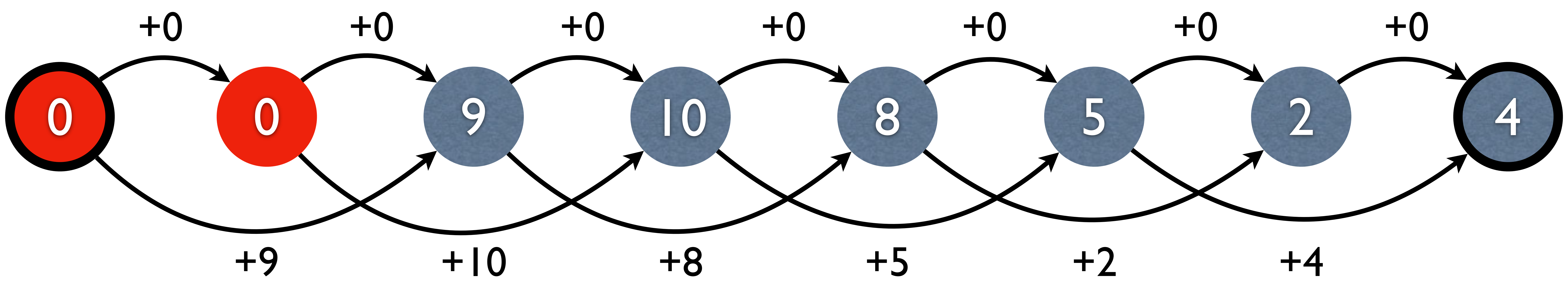
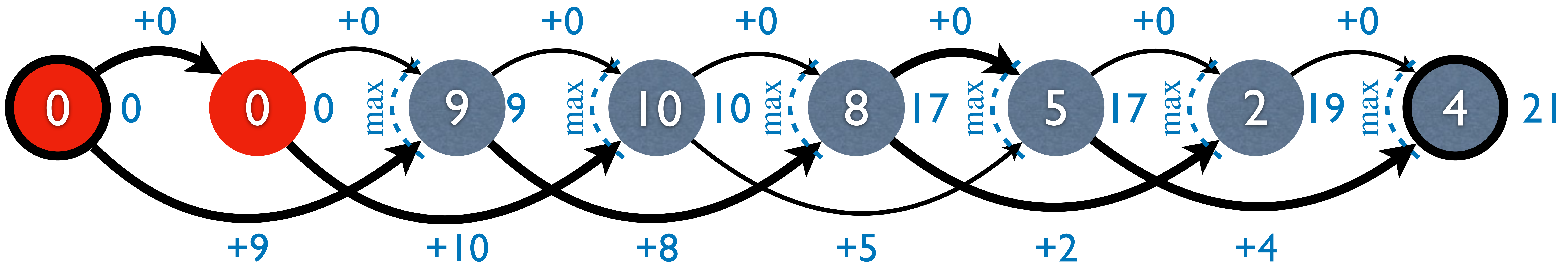
  - $f(i)$: longest path between source and node i
- fibonacci & bitstrings: number of paths between source and target





Summary

- Divide-and-Conquer = divide + conquer + combine
- Dynamic Programming = multiple divides + memoized conquer + summarized combine
- two implementation styles
  - 1. recursive top-down + memoization
  - 2. bottom-up
- backtracking to recover best solution for optimization problems
  - 1. backpointers (recommended); 2. store subsolutions (not recommended — often slows down); 3. recompute on-the-fly
- two operators: $\oplus$ for summary (across multiple divides) and $\otimes$ for combine (within a divide)
- counting problems vs. optimization problems ("cost-reward model")
- three steps in solving a DP problem
  - define the subproblem
  - recursive formula
  - base cases

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  - define the subproblem
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  - base cases

cost reward

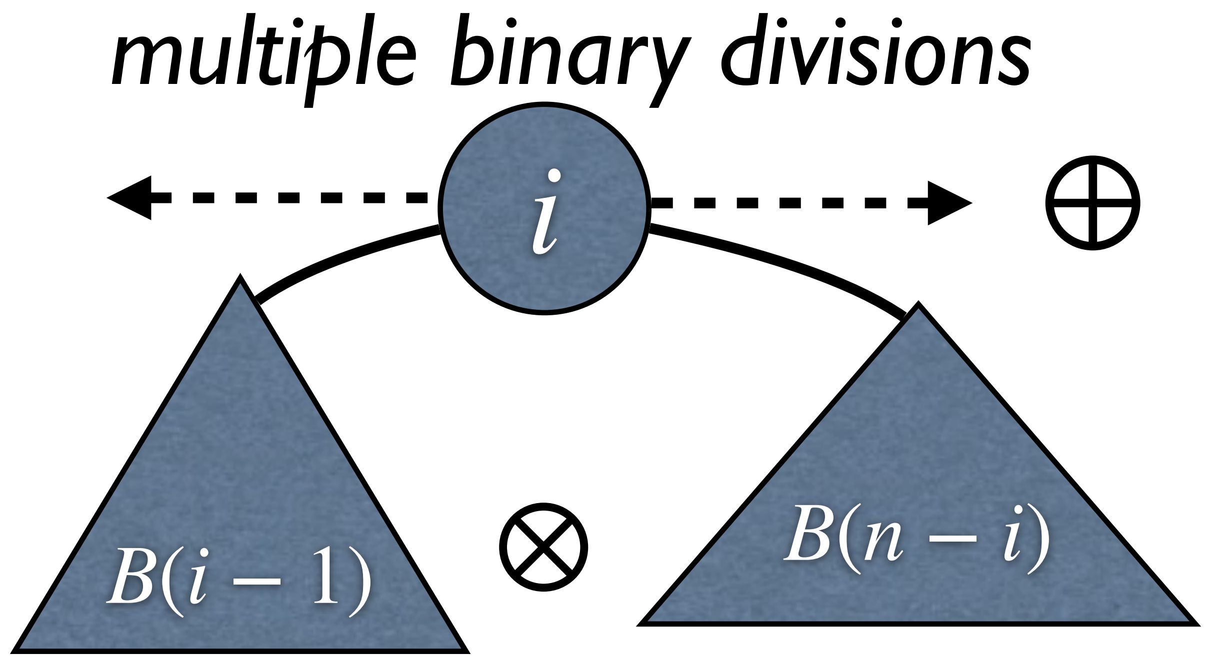
$$f(n) = \max \begin{cases} f(n-1) + 0 \\ f(n-2) + a[n] \end{cases}$$

summary operator $\oplus$ (across divides)

combination operator $\otimes$ (within a divide)

Deeper Understanding of DP

- divide-n-conquer
  - single division, independent conquer, combine
- DP = divide-n-conquer with multiple divisions
  - for each possible division
    - divide
    - conquer with memoization
    - combine subsolutions using the combination operator $\otimes$
  - summarize over all possible divisions using the summary operator $\oplus$
- multiple divisions => overlapping subproblems
  - each single division => independent subproblems!



	$\oplus$	$\otimes$
Fib	+	x
MIS	max	+
# BSTs	+	x
knapsack	max	+
shortest path	min	+

$$B(n) = \oplus_{i=1}^{n} \left(B(i-1) \otimes B(n-i) \right)$$

$$B(0) = 1$$

Unary vs. Binary Divisions

$(b): T(n) = T(n-1) + \ldots$

$(a): T(n) = 2T(n/2) + \ldots$

$(c): T(n) = T(n/2) + \ldots$

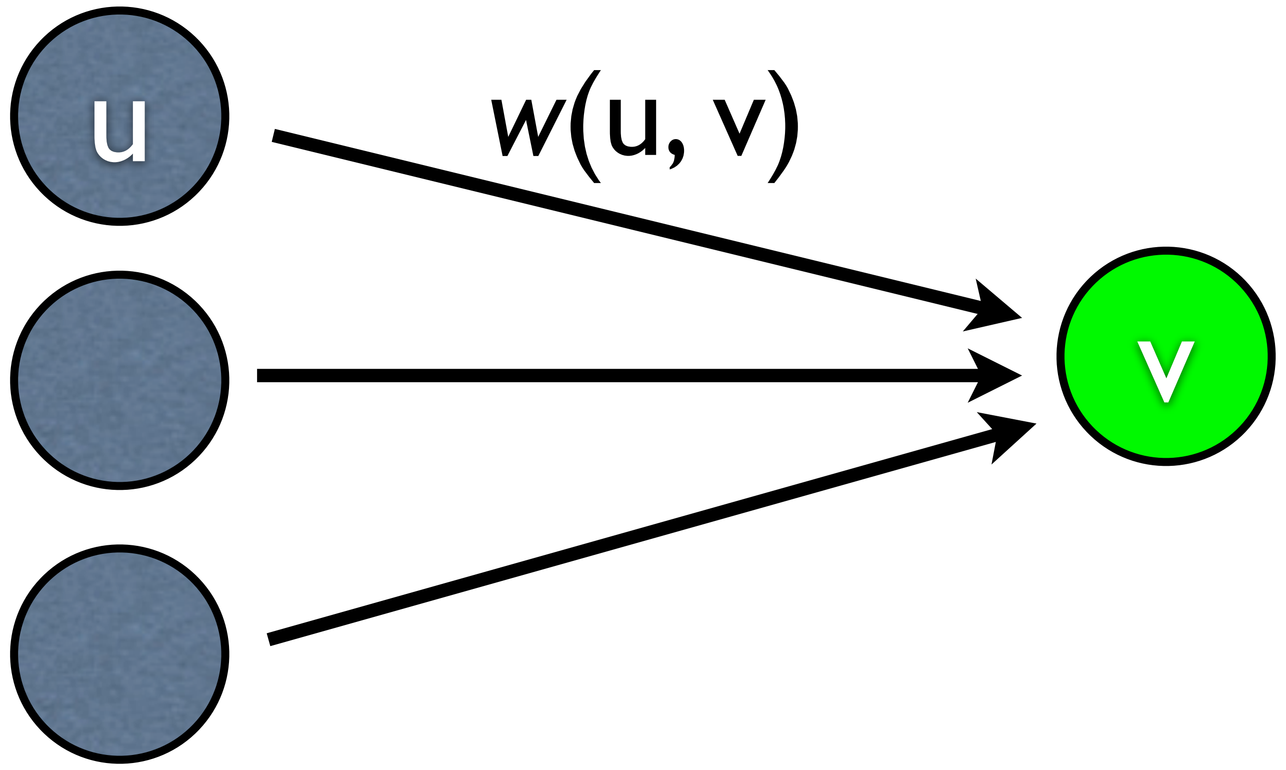
	branching (binary division)	one-sided (unary division)
divide-n-conquer	quicksort, best-case	quicksort, worst-case (*b*)
	mergesort	quickselect: worst (*b*), best (*c*)
	(balanced) tree traversal (DFS)	binary search: (*c*)
	heapify (top-down)	search in BST: worst (*b*), best (*c*)
DP	# of BSTs (hw5), *midterm*	Fib, # of bitstrings (hw5)…
	optimal BST, *final*	max indep. set (hw5)
	RNA folding (hw10)	knapsack (hw6), *midterm*
	context-free parsing	Viterbi (hw8), *final*
	matrix-chain multiplication, …	LCS, LIS, edit-distance,…

Two Divisions vs. Multiple Divisions (# of Choices)

	two divisions	multiple division
DP	Fib, # of bitstrings (hw5)…	# of BSTs (hw5)
	max indep. set (hw5)	unbounded knapsack (hw6)
	0-1 knapsack (hw6)	bounded knapsack (hw6)
		Viterbi (hw8)
		RNA folding (hw10)

Viterbi Algorithm for DAGs

1. topological sort
2. visit each vertex v in sorted order and do updates
   - for each incoming edge (u, v) in E
   - use d(u) to update d(v): $d(v) \oplus = d(u) \otimes w(u, v)$
   - key observation: d(u) is fixed to optimal at this time



- time complexity: O(V + E)

Variant 1: forward-update

1. topological sort
2. visit each vertex v in sorted order and do updates
   - for each outgoing edge (v, u) in E
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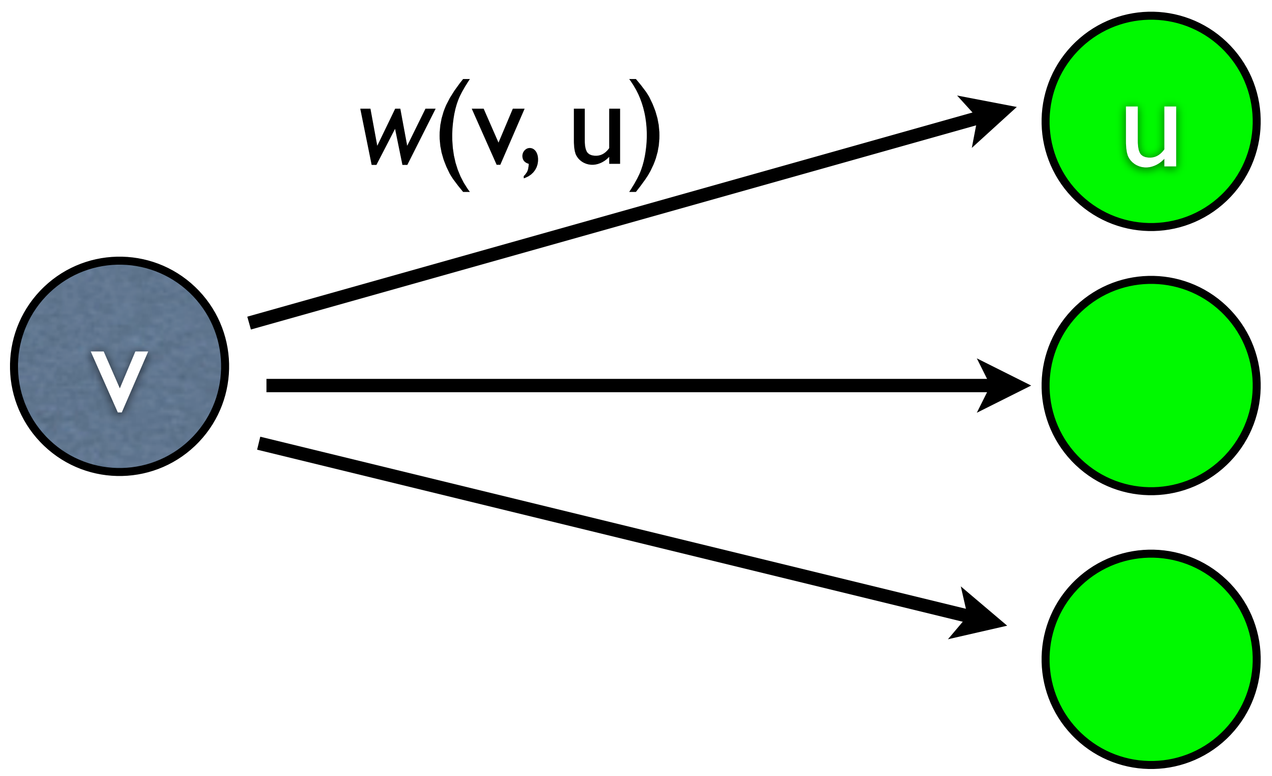
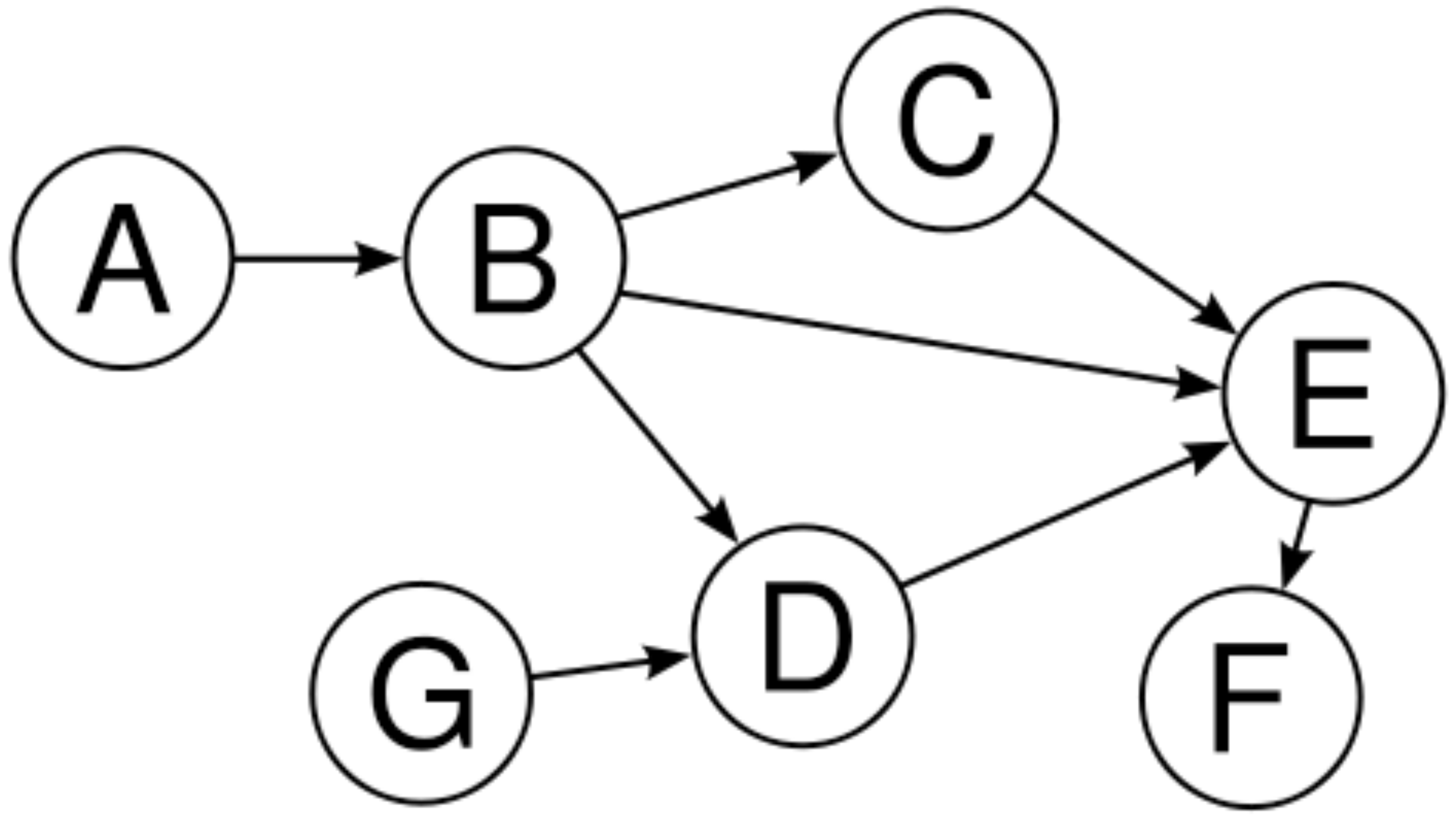


- time complexity: O(V + E)

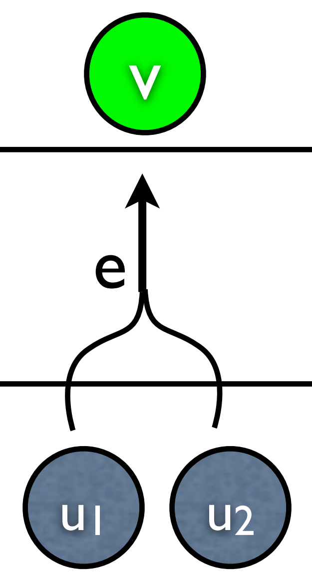
Variant 2: Recursive Descent

- Top-down Recursion + Memoization = Bottom-up
- Start from the target vertex, going backwards
  - remember each visited vertex
- Sometimes easier to implement
- There is a tradeoff b/w top-down and bottom-up

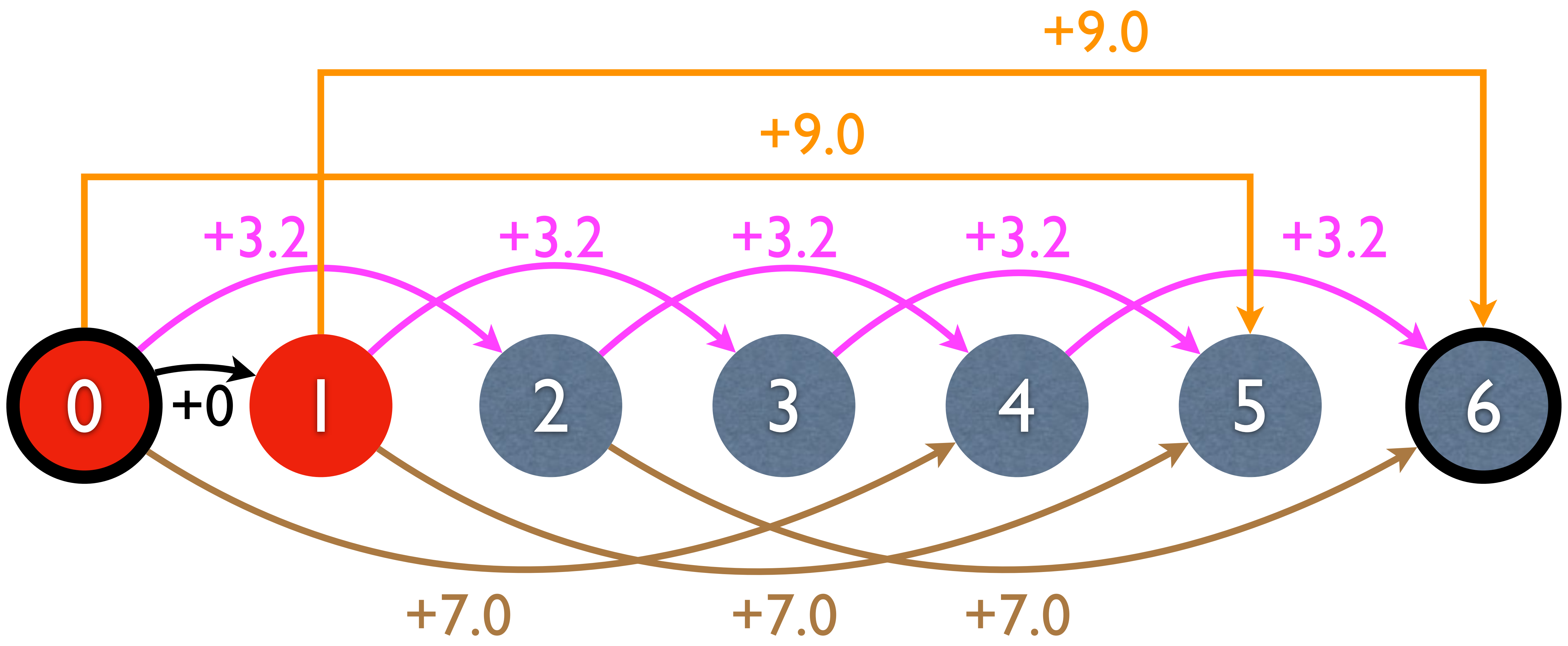


One-way vs. Two-way Divides (Graph vs. Hypergraph)

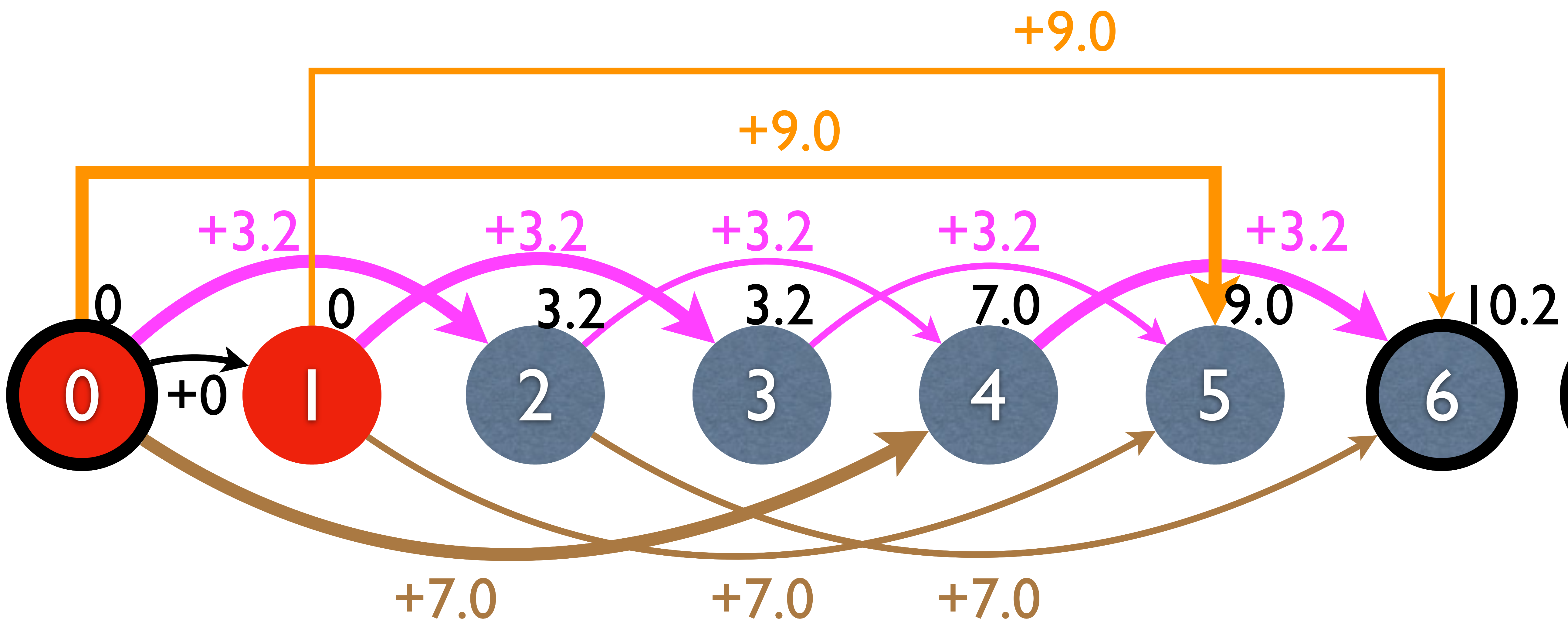
		two-way (binary divide)		one-way (unary divide)
divide-n-conquer	binary tree	quicksort, best-case	linear-chain	quicksort, worst-case
		mergesort		quickselect
		tree traversal (DFS)		binary search
		heapify (top-down)		search in BST
DP	hypergraph	# of BSTs (hw5)	graph	Fib, # of bitstrings (hw5)…
		optimal BST		max indep. set (hw5)
		RNA folding (hw10)		knapsack (all kinds, hw6)
		context-free parsing		Viterbi (hw8)
		matrix-chain multiplication, …		LCS, LIS, edit-distance,…

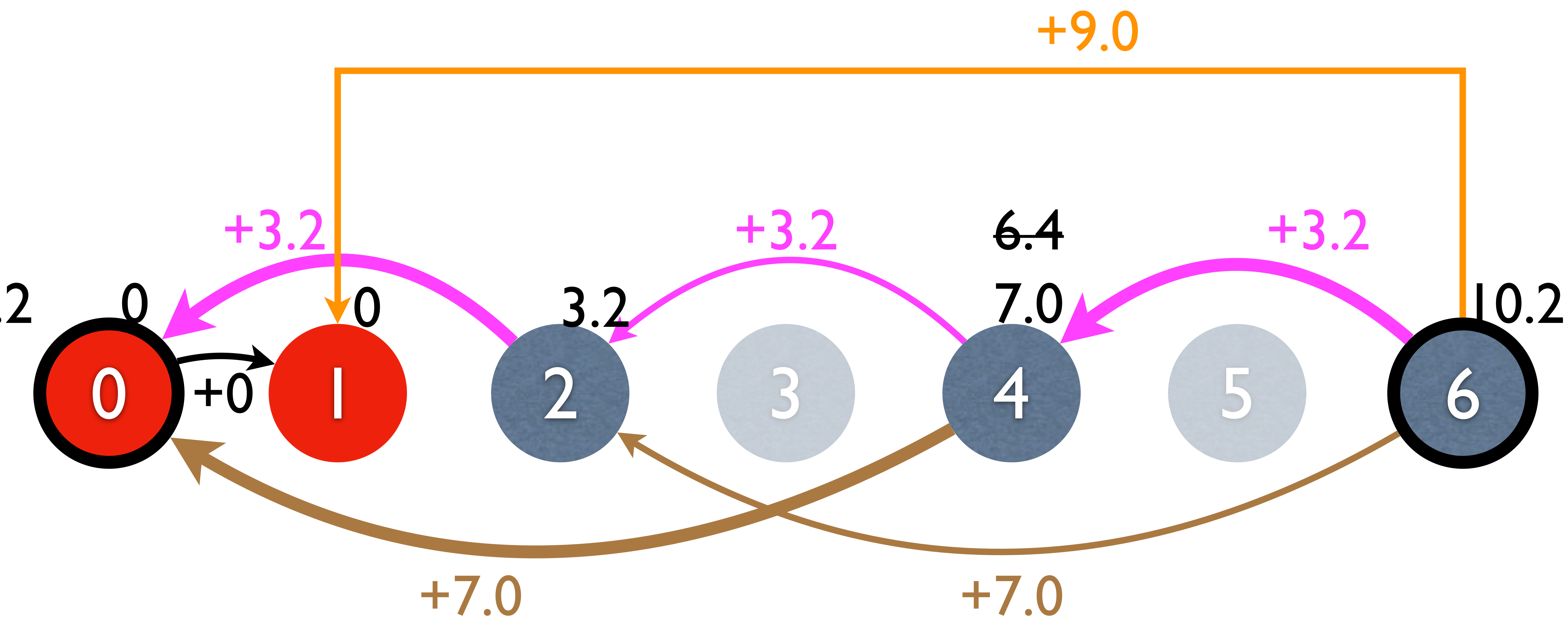


Graph Interpretation of Unbounded Knapsack



i	*w_i*	*v_i*
1	2	3.2
2	5	9.0
3	4	7.0





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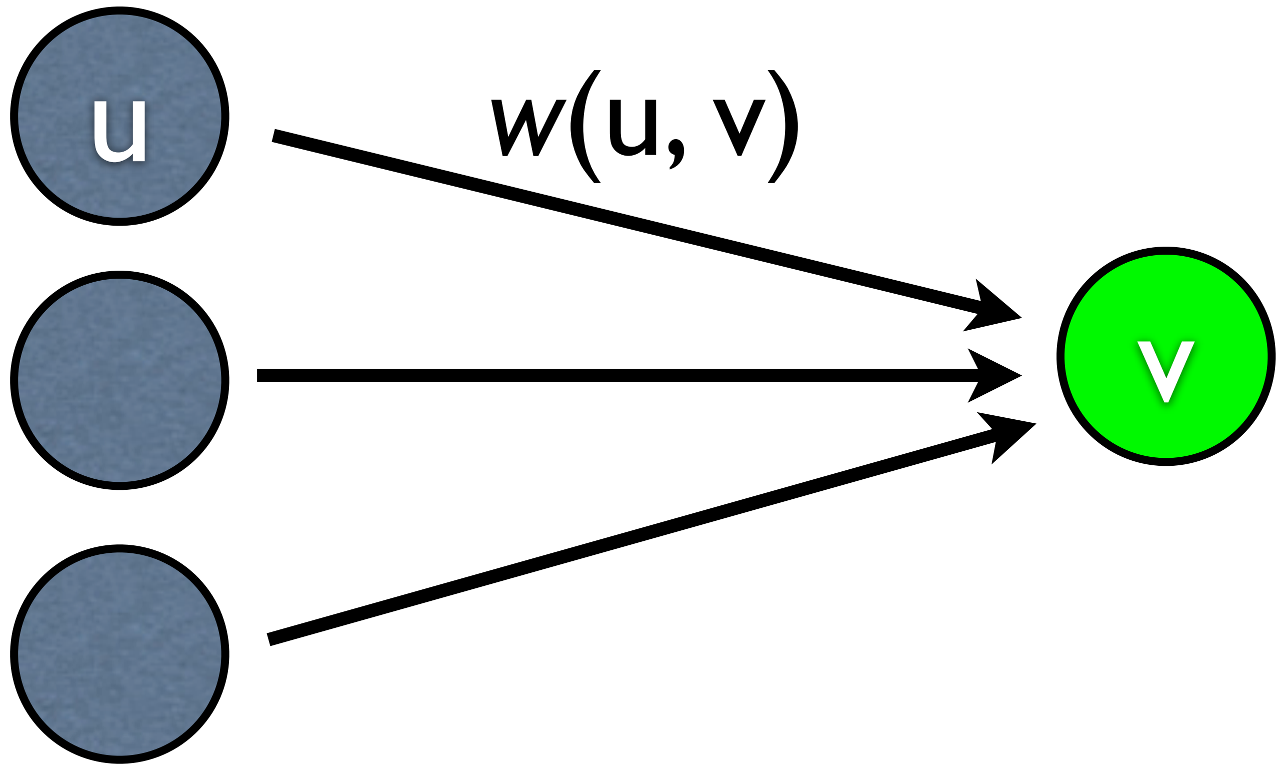
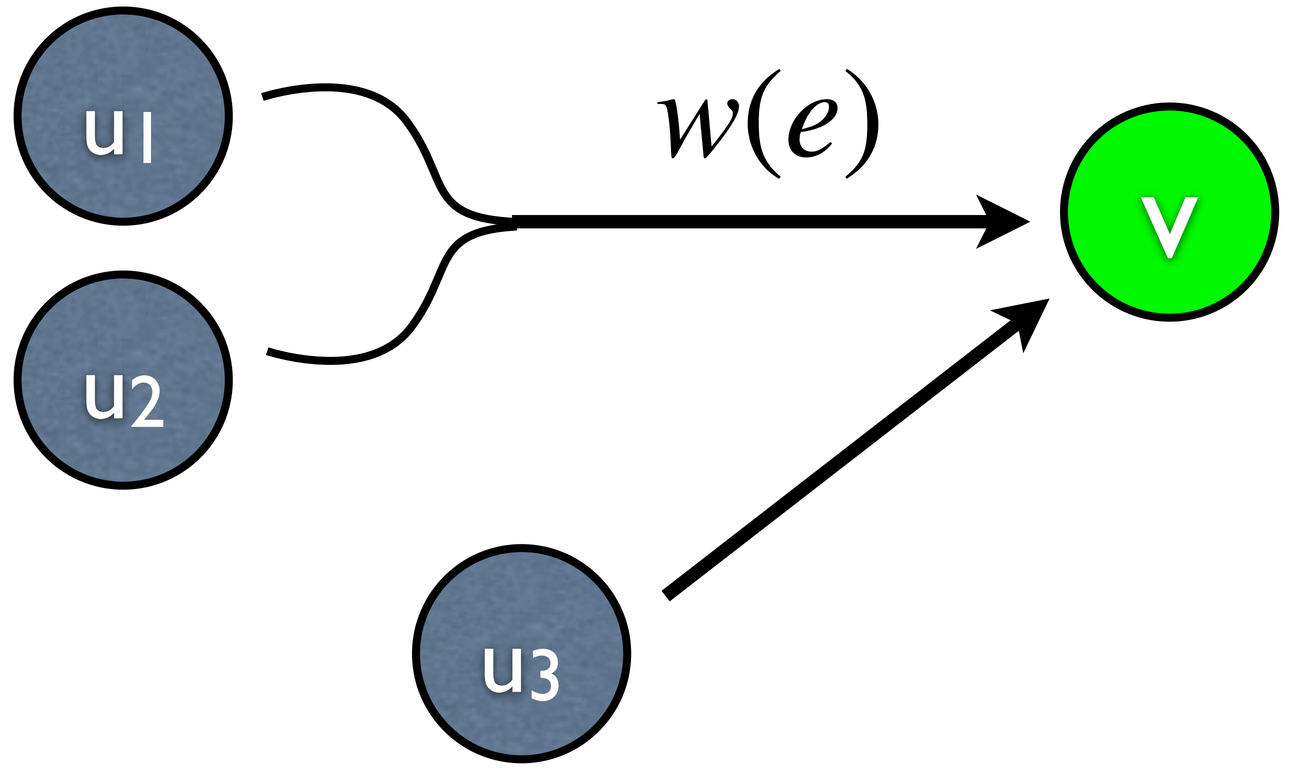


- time complexity: O(V + E)

Generalized Viterbi for DAHs (Hypergraphs)

1. topological sort
2. visit each vertex v in sorted order and do updates
   - for each incoming hyperedge e = ((u_1, .., $u_{|e|}$), v, w(e))
   - use d(u_i)'s to update d(v)
   - key observation: d(u_i)'s are fixed to optimal at this time

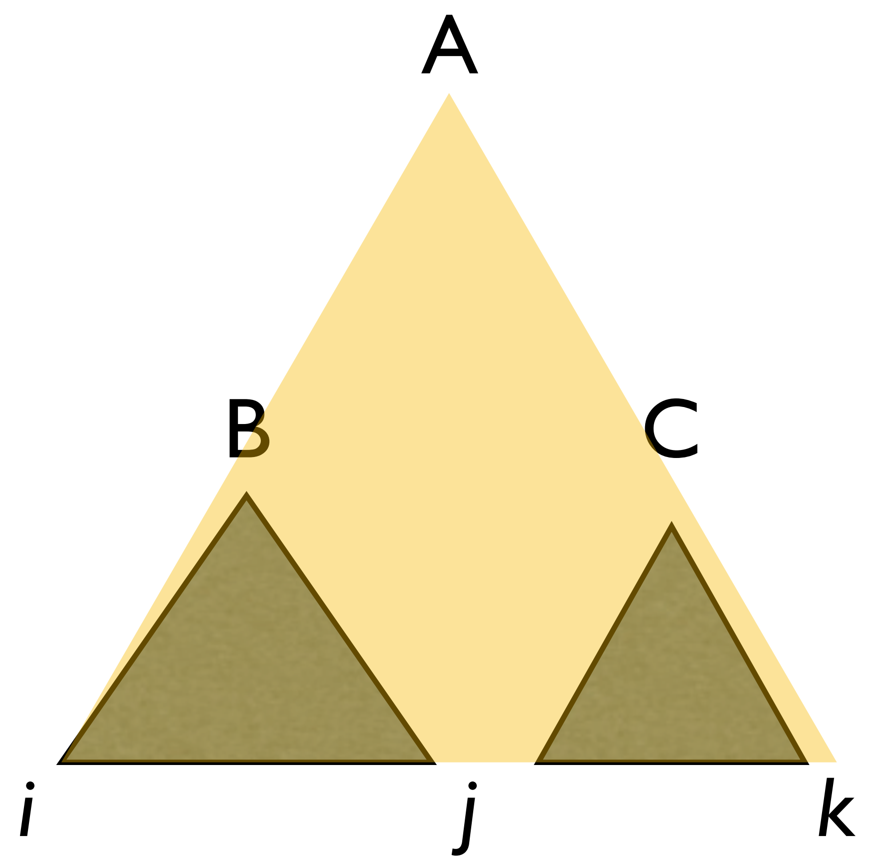


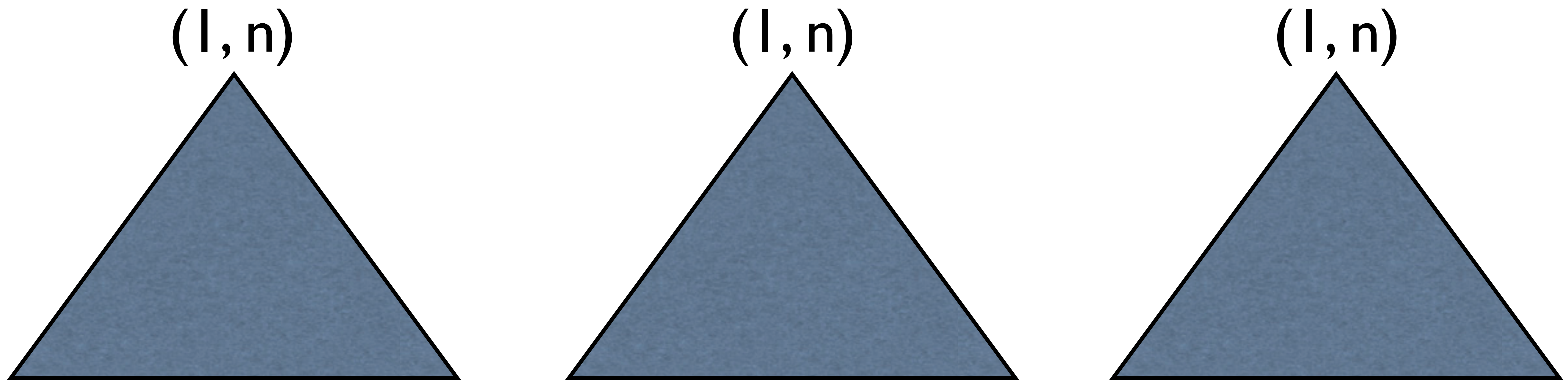
$$d(v) \oplus = d(u_1) \otimes d(u_2) \otimes w(e)$$

- time complexity: O(V + E) (assuming constant arity)

Example: RNA Folding and CKY Parsing

- typical instance of the generalized Viterbi for DAHs
- many variants of CKY ~ various topological ordering
- Nussinov algorithm in RNA is almost identical to CKY but w/o overcounting





all $O(n^3)$

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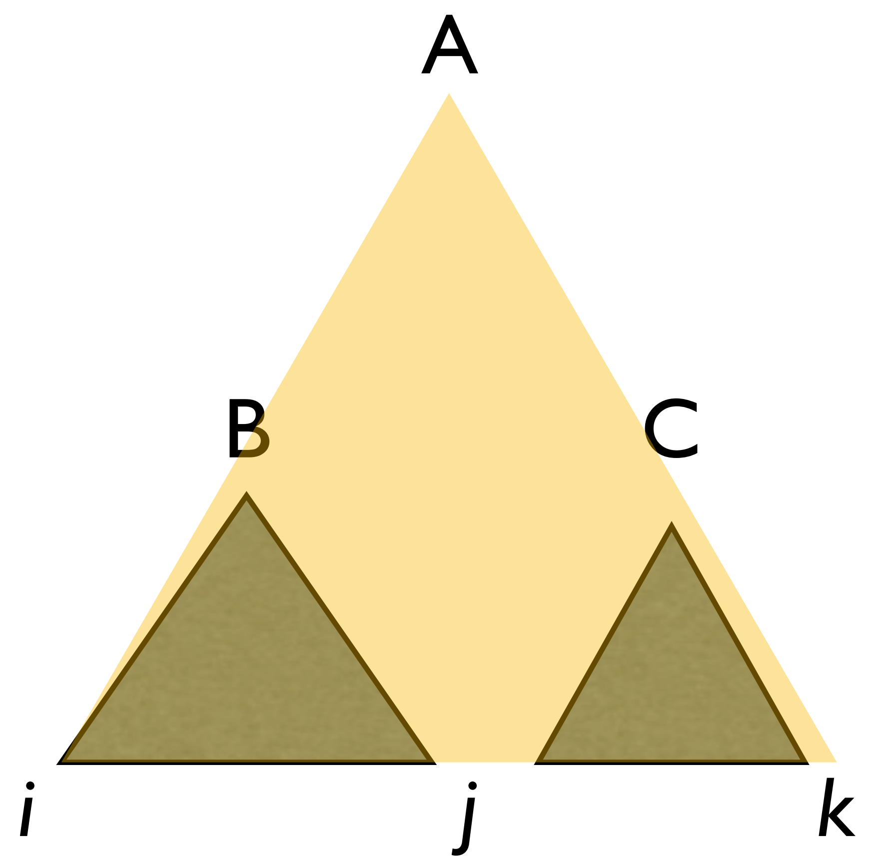

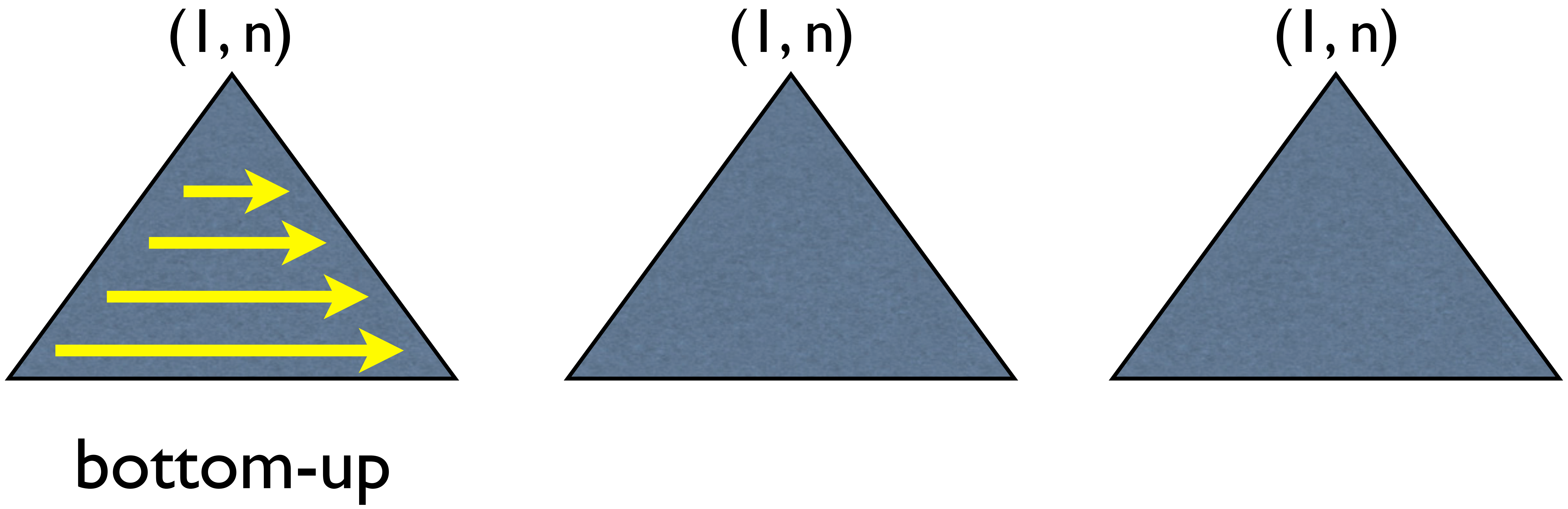

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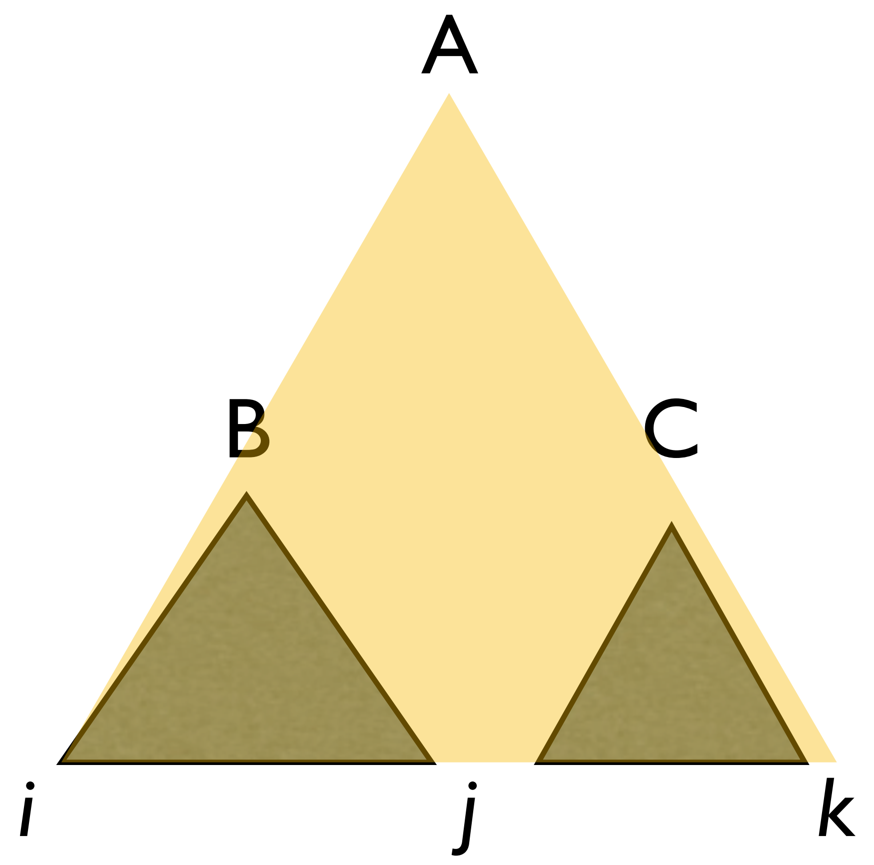

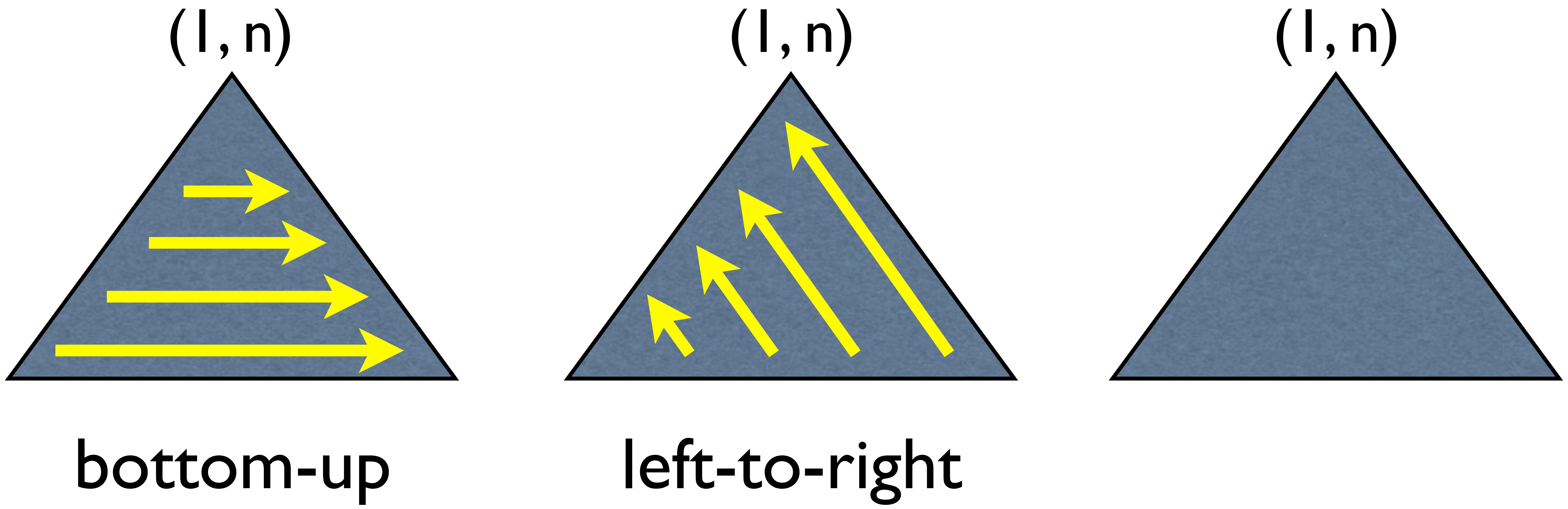

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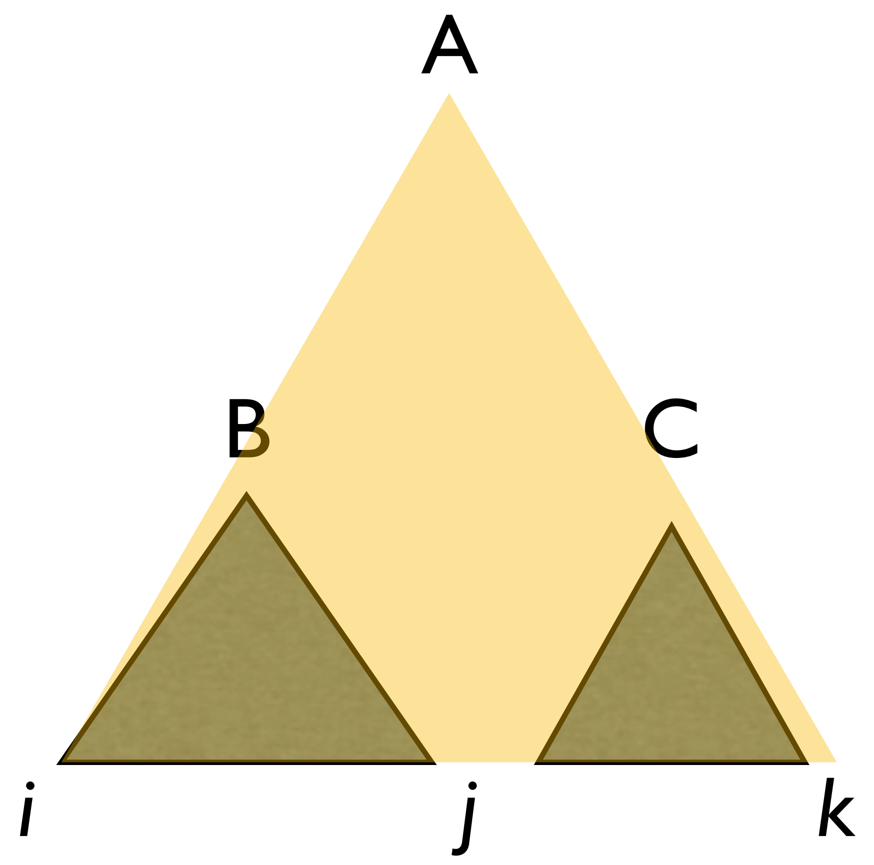
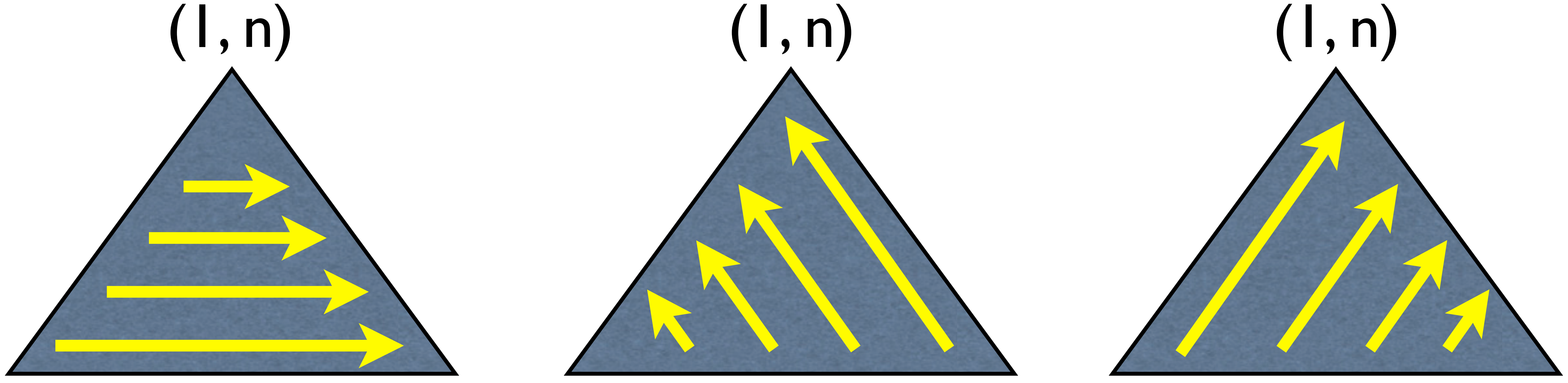


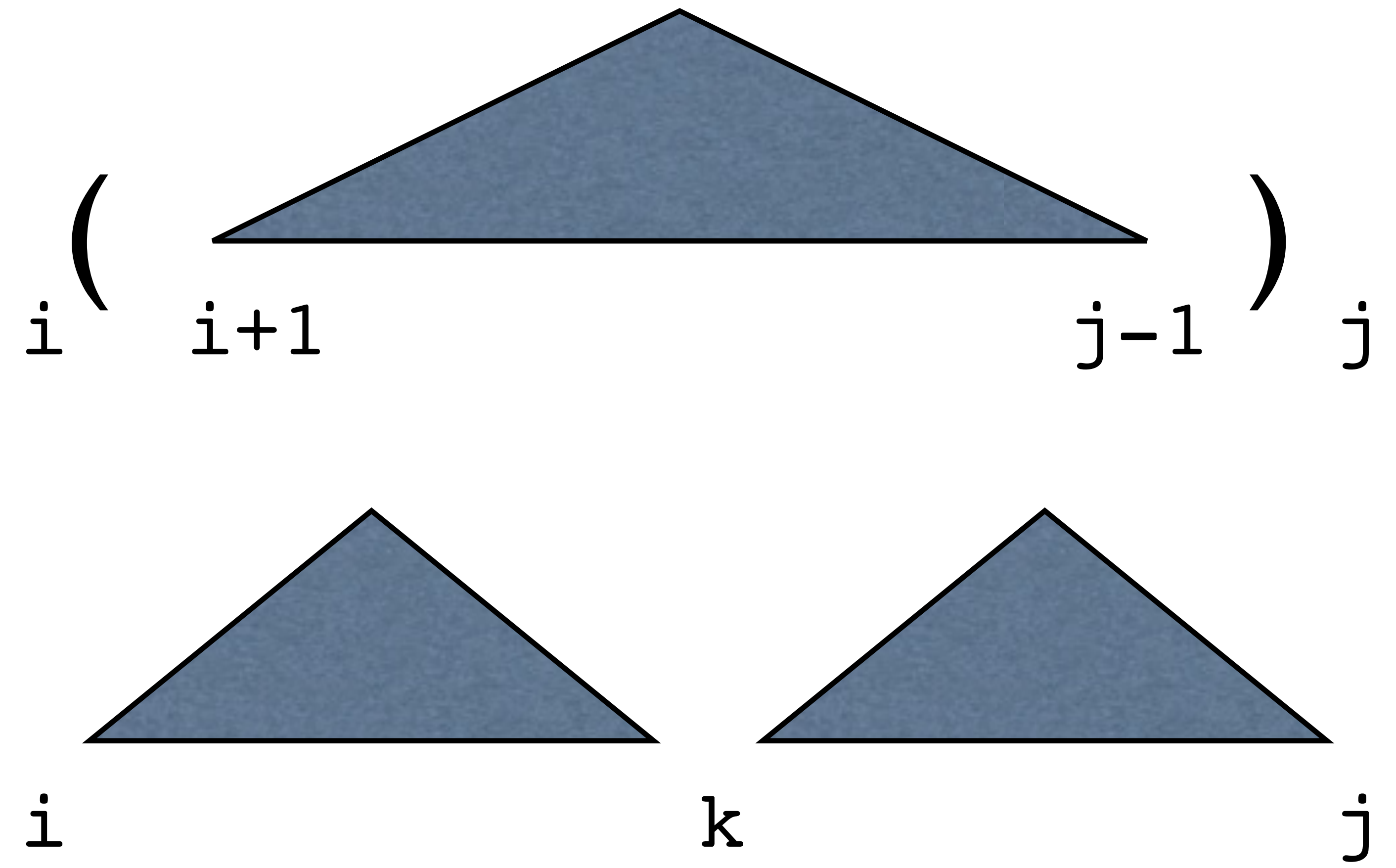


bottom-up　　left-to-right　　right-to-left

all $O(n^3)$

Example: RNA Folding as CKY Parsing

- Dynamic Programming — $O(n^3)$
  - bottom-up CKY parsing
  - example: maximize # of pairs (A-U, G-C, or G-U)



A C A G U

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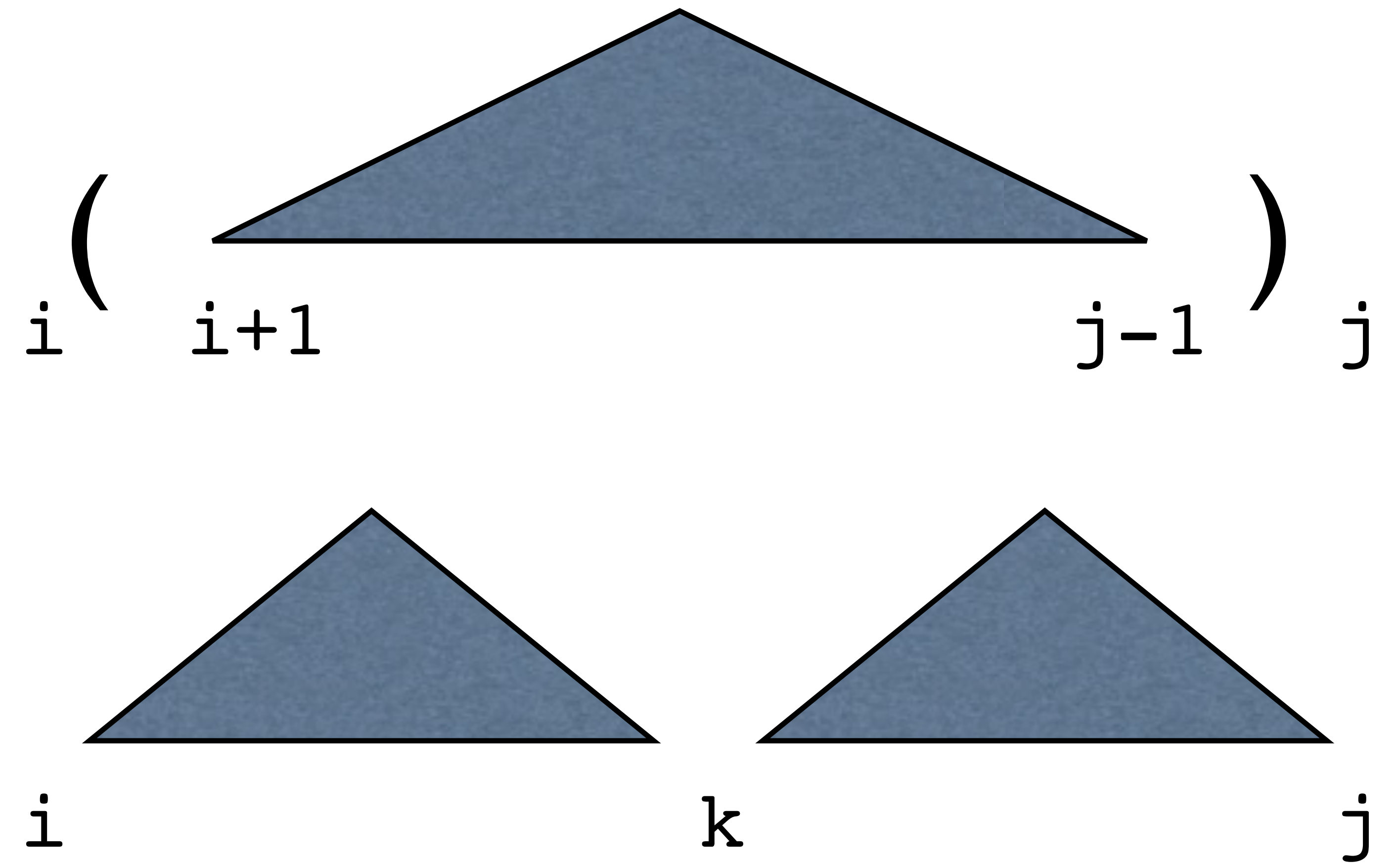

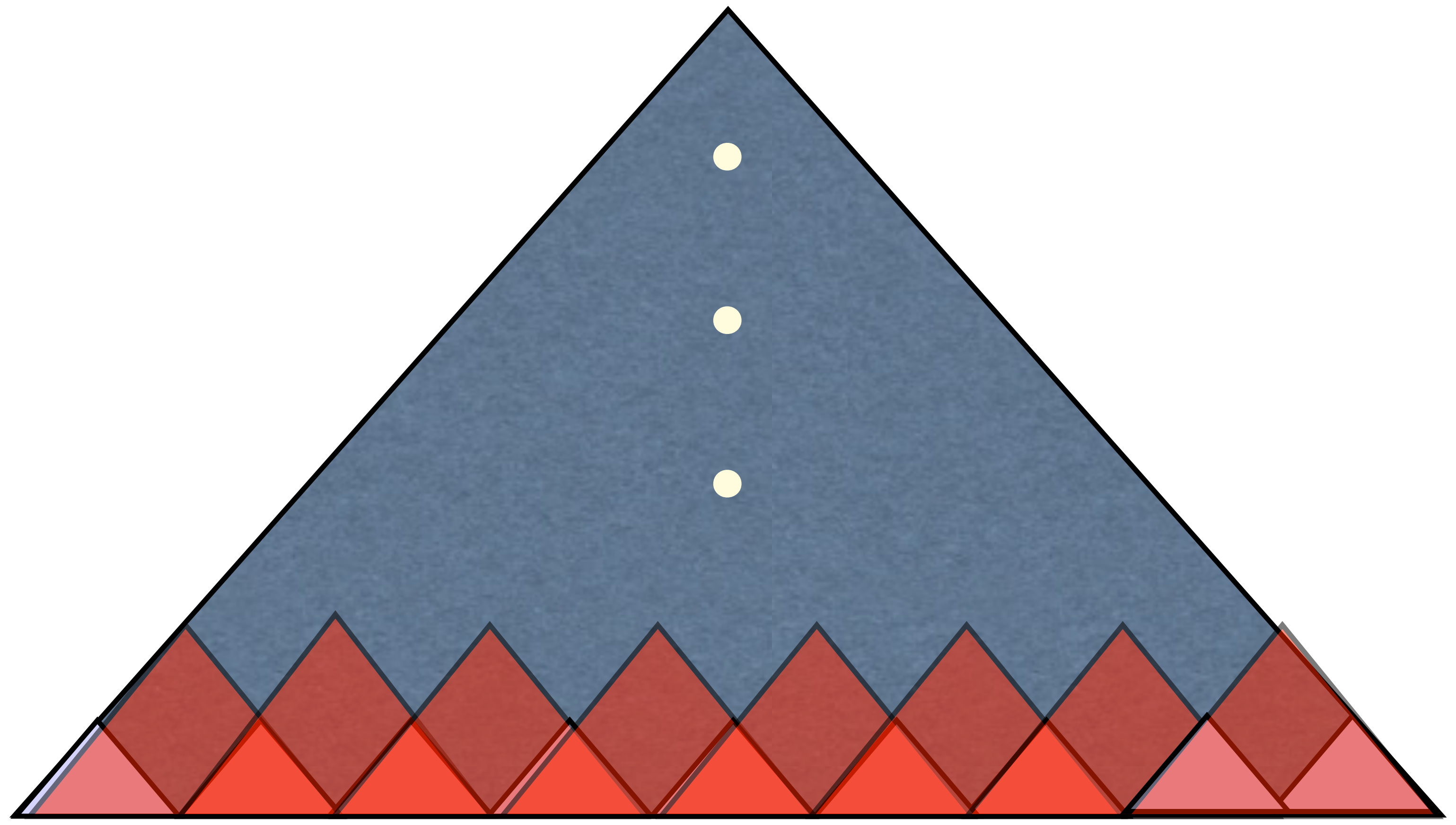

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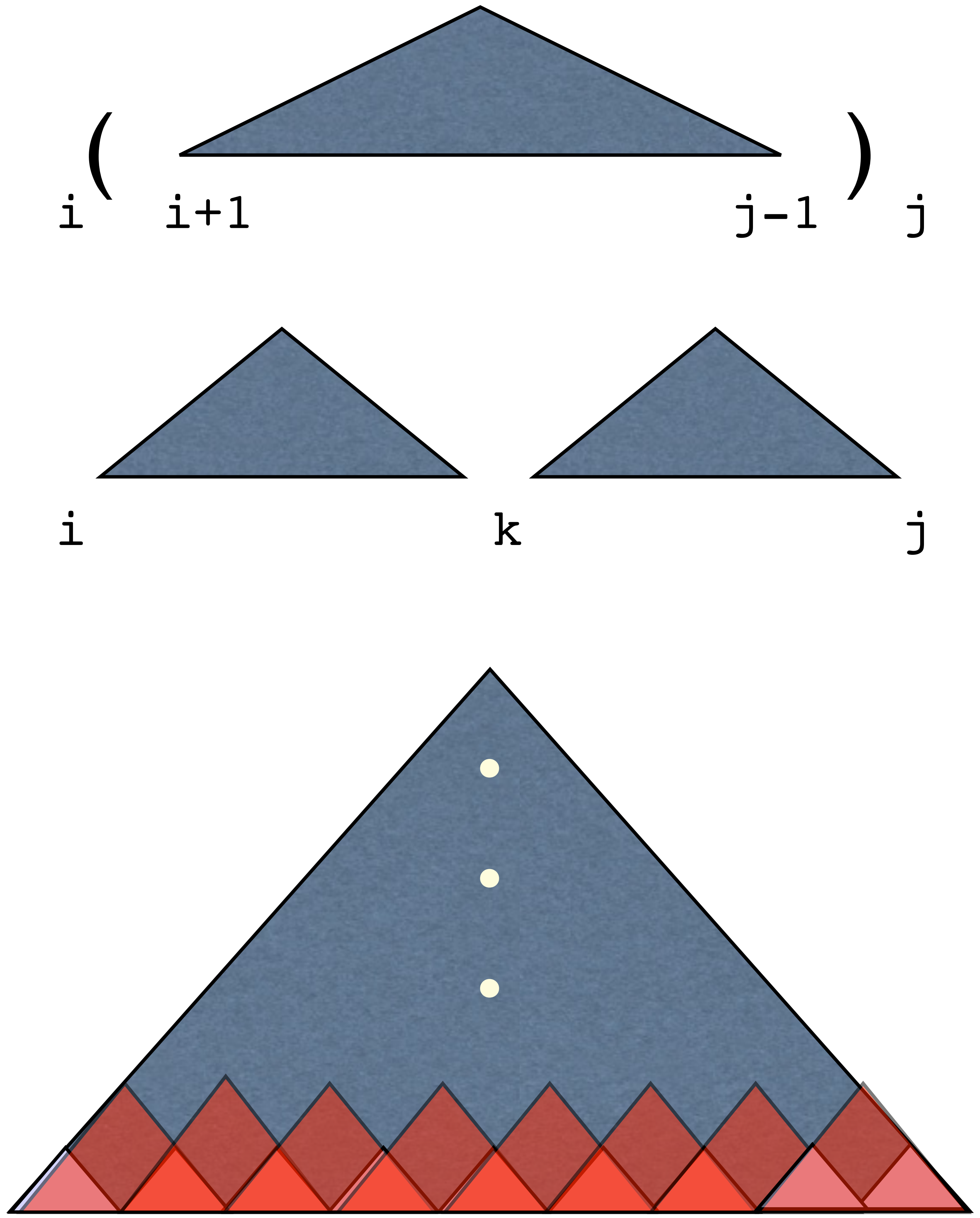

.   .   .   .   .

A   C   A   G   U

Example: RNA Folding as CKY Parsing

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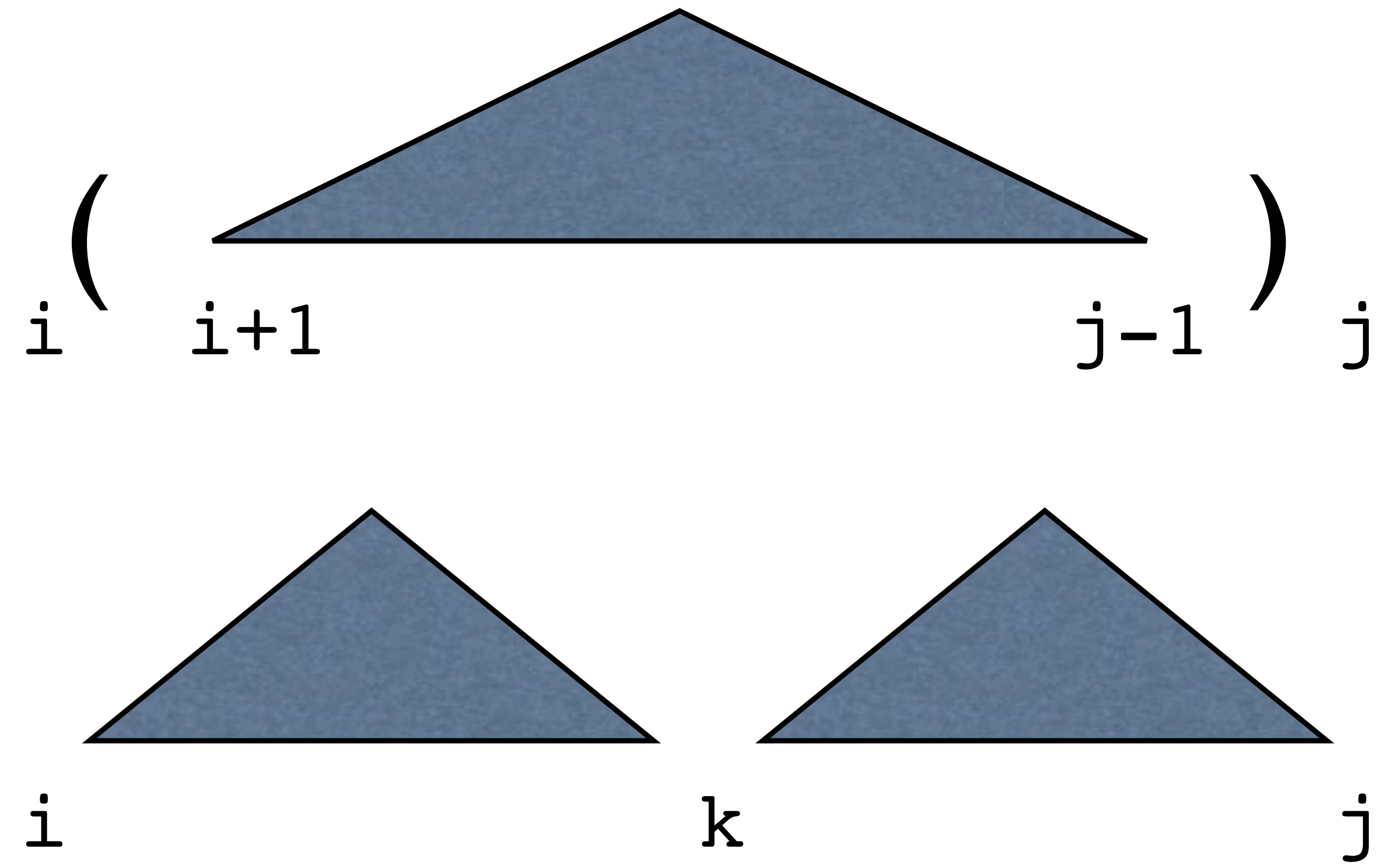

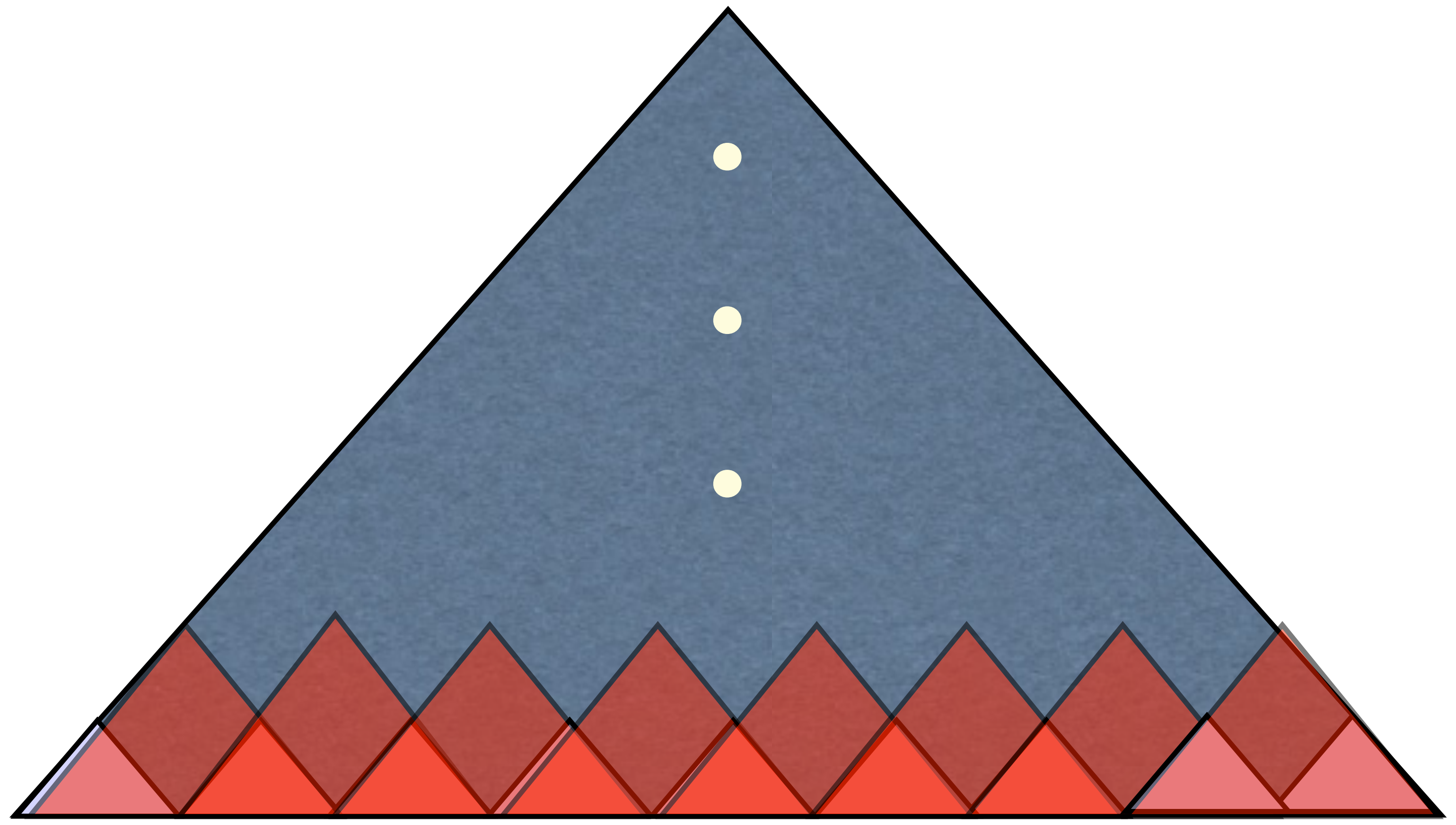

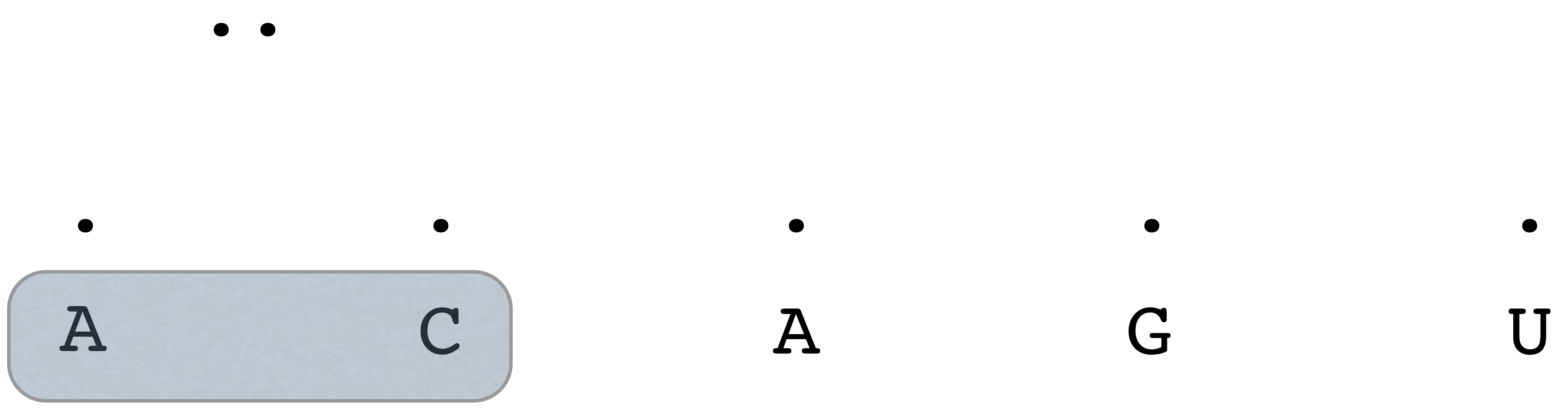

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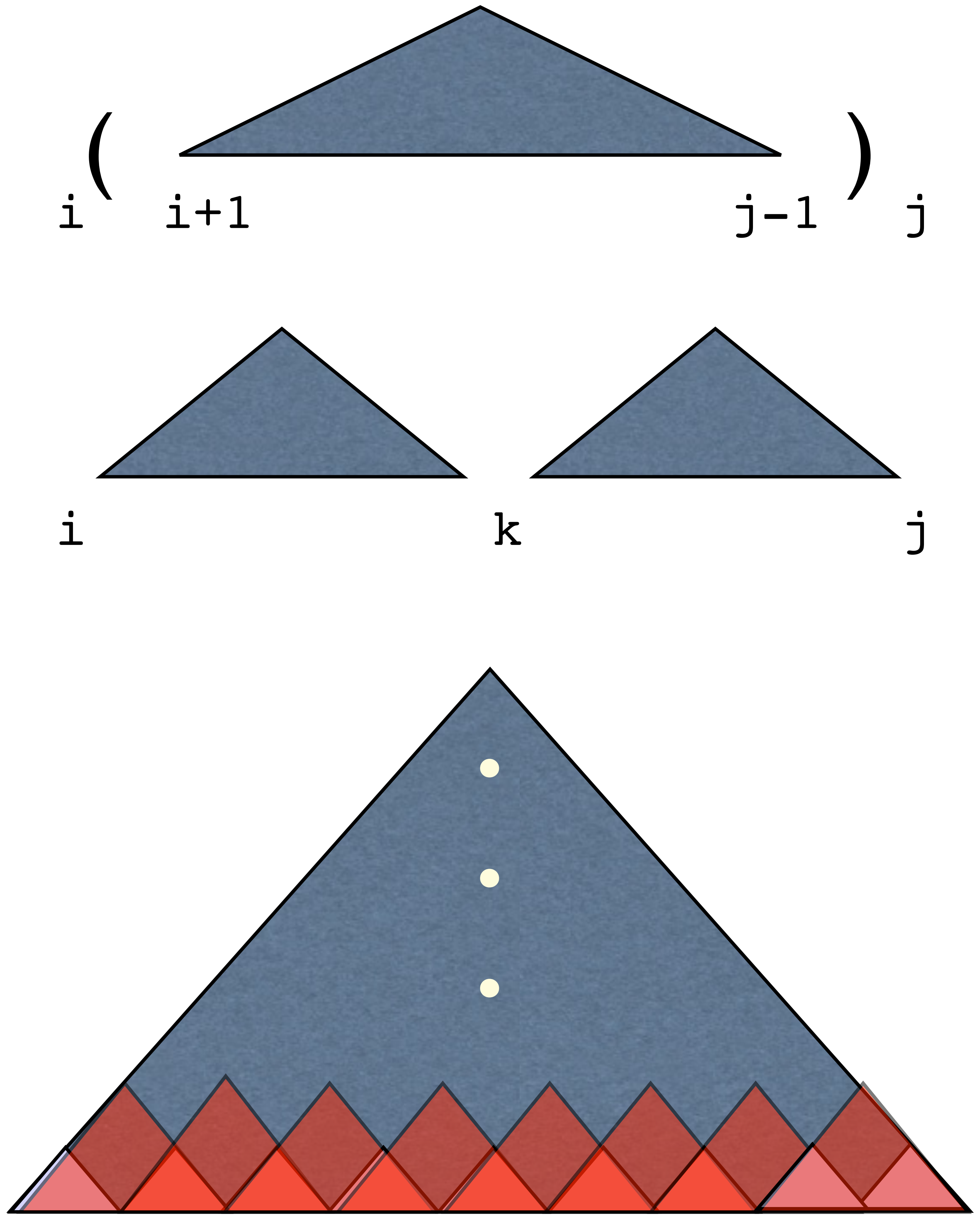

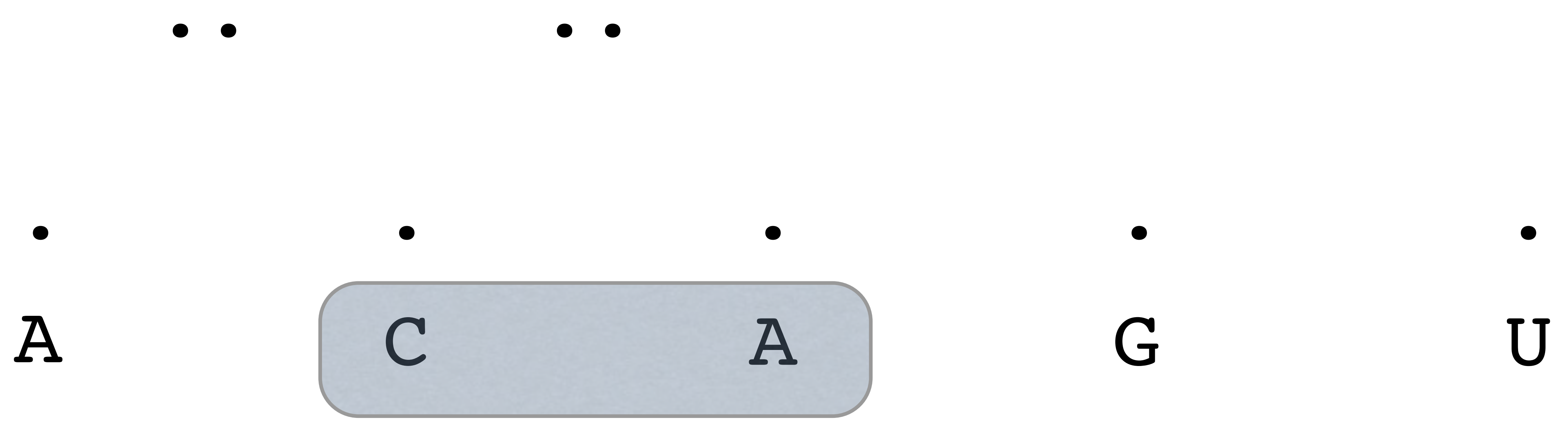

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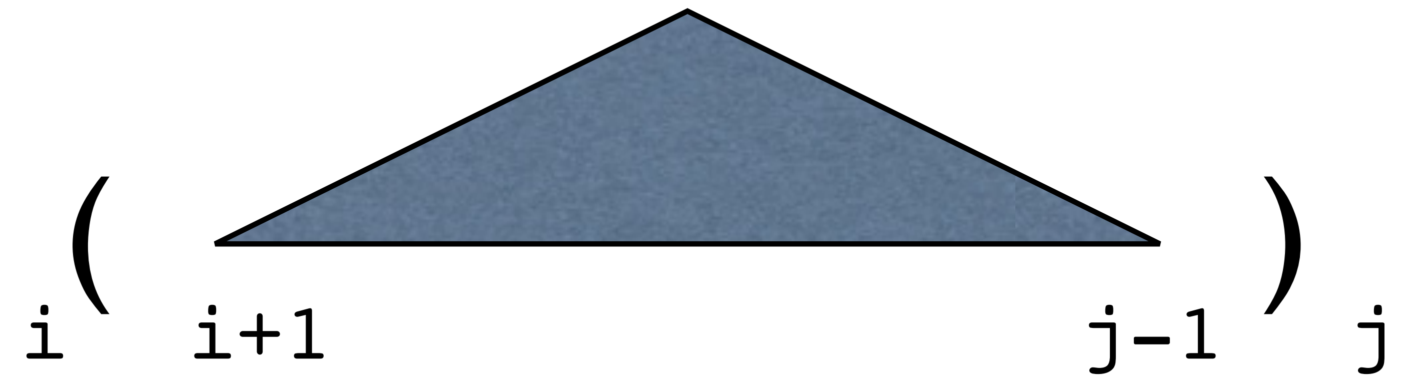

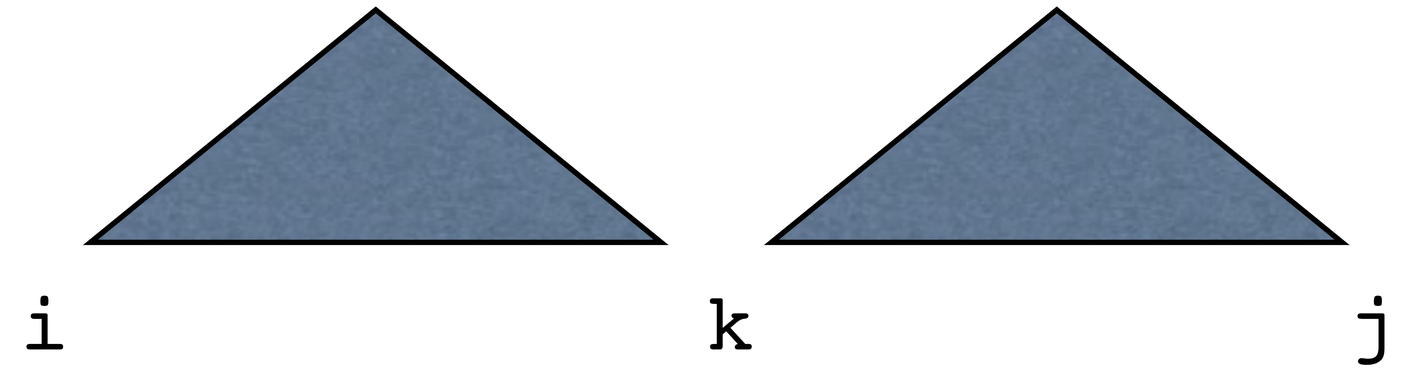

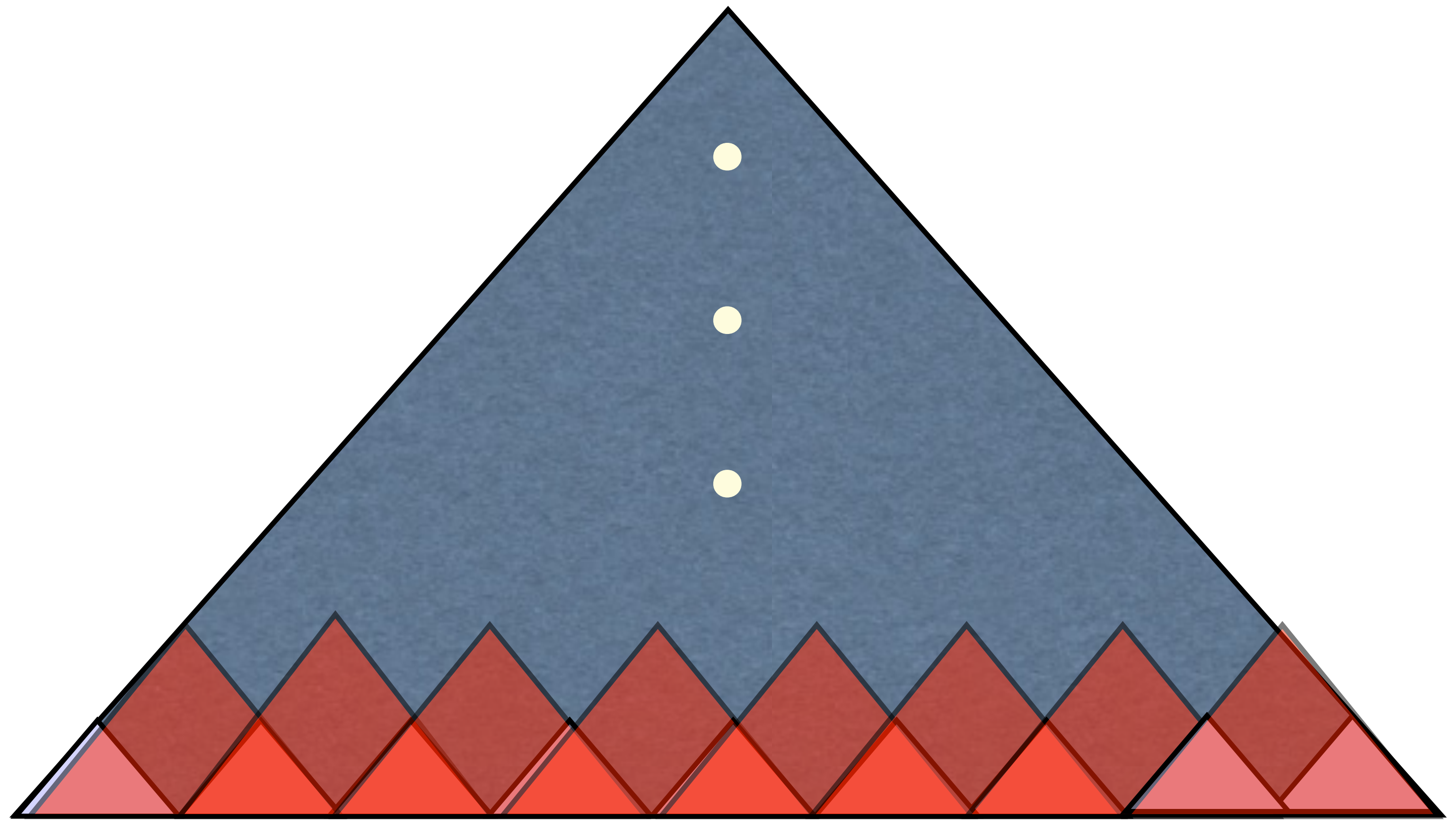

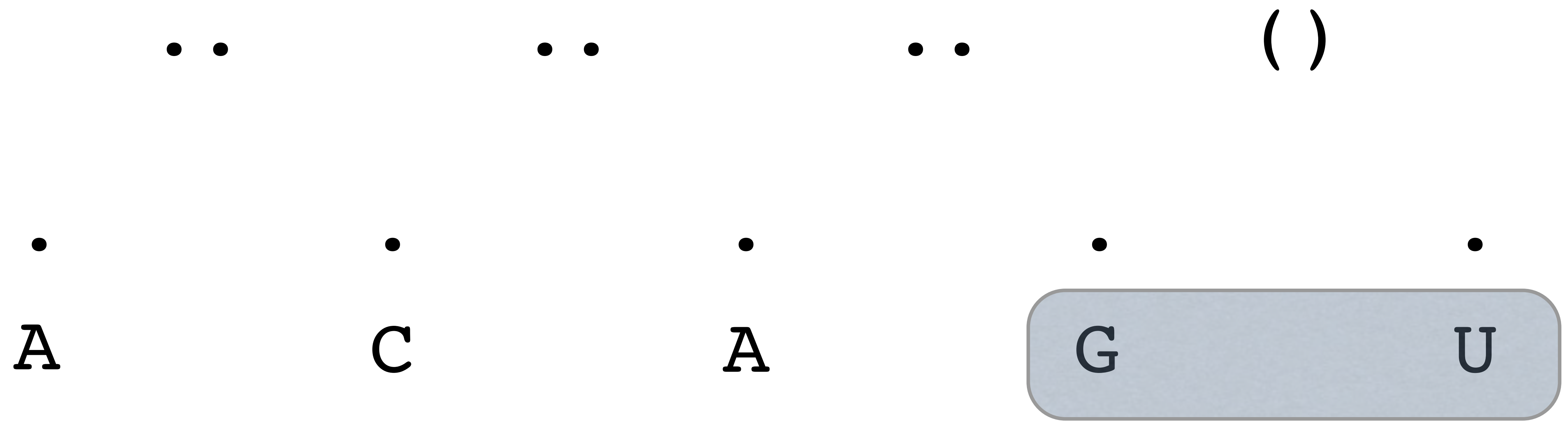

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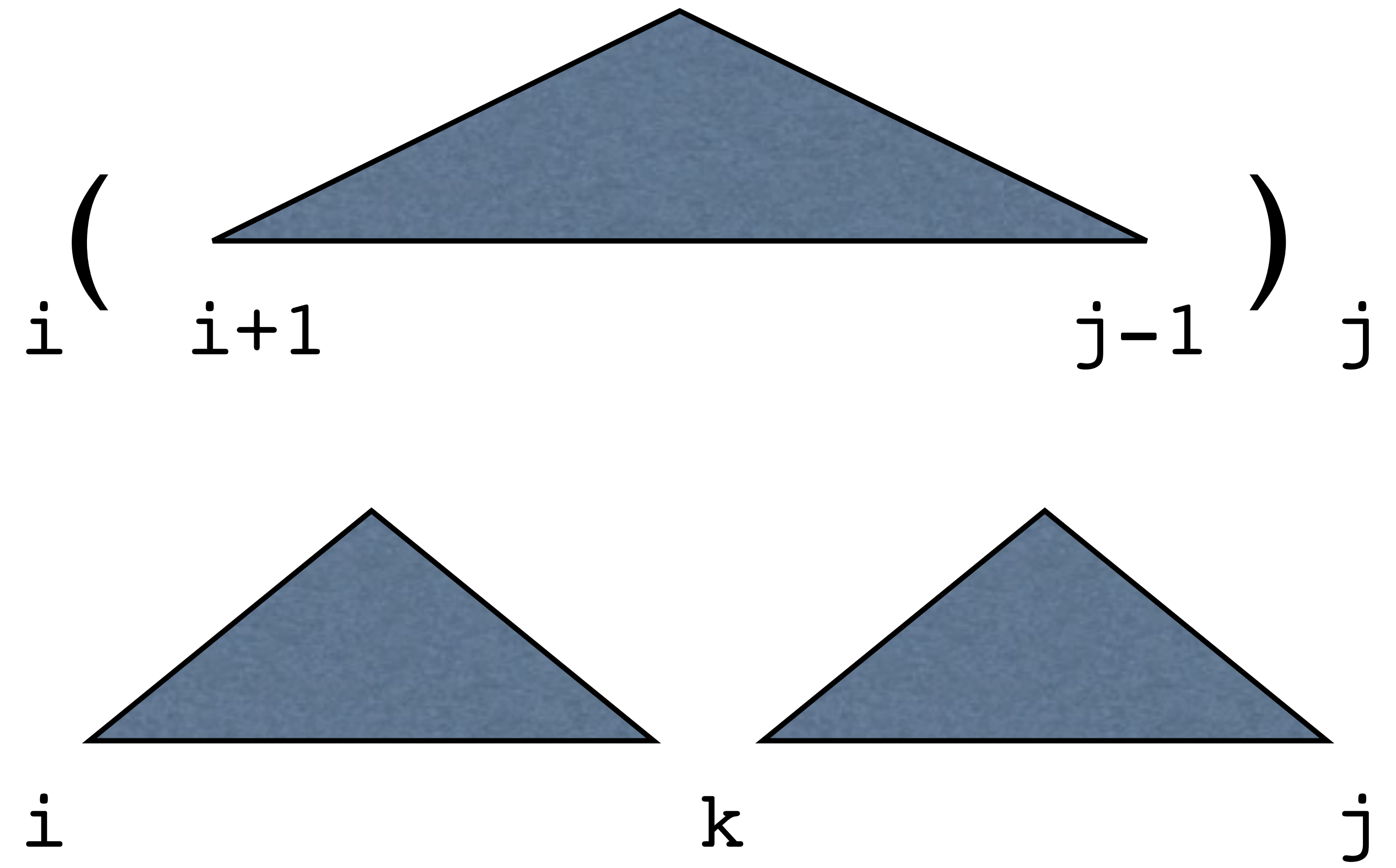

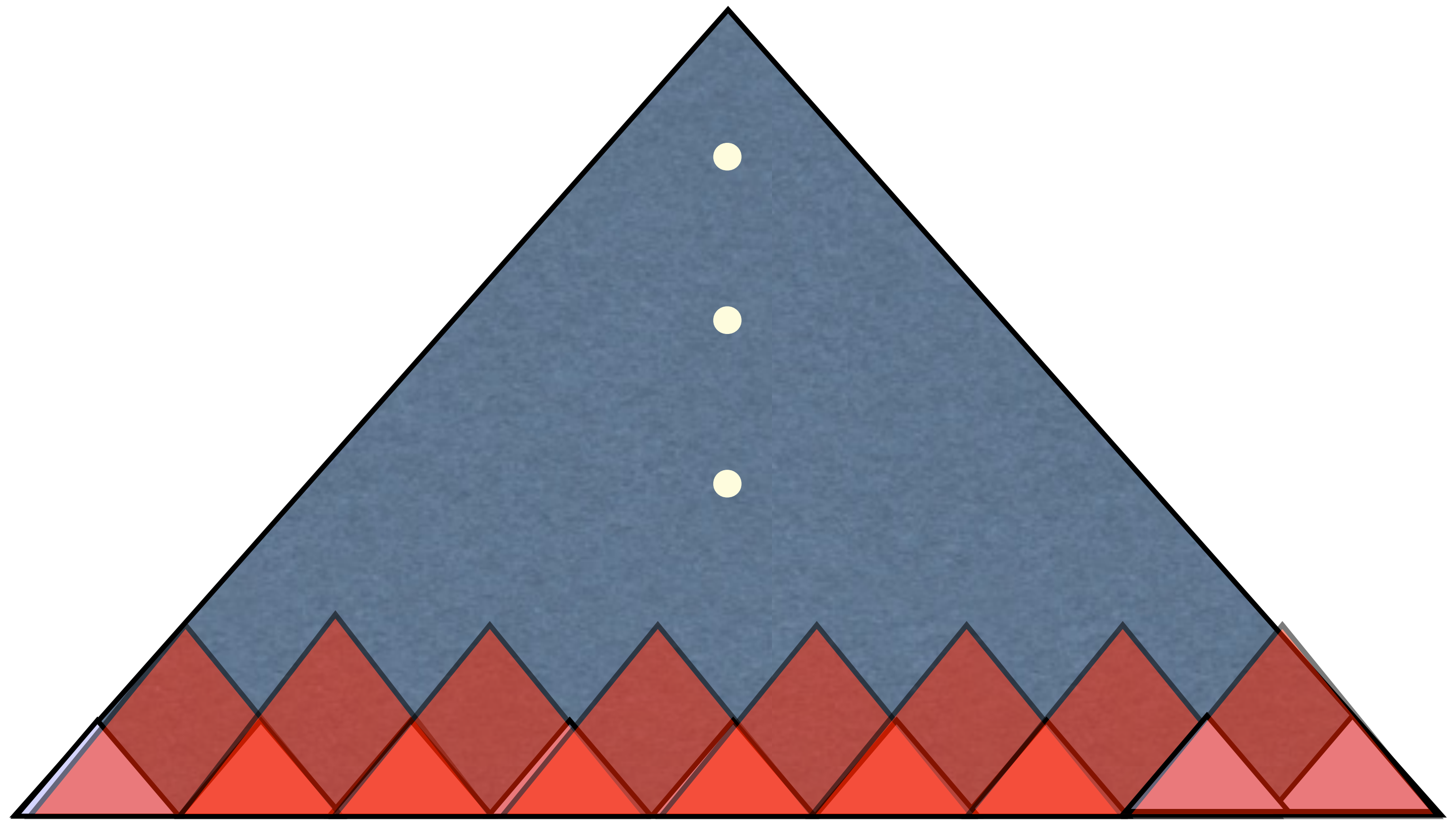

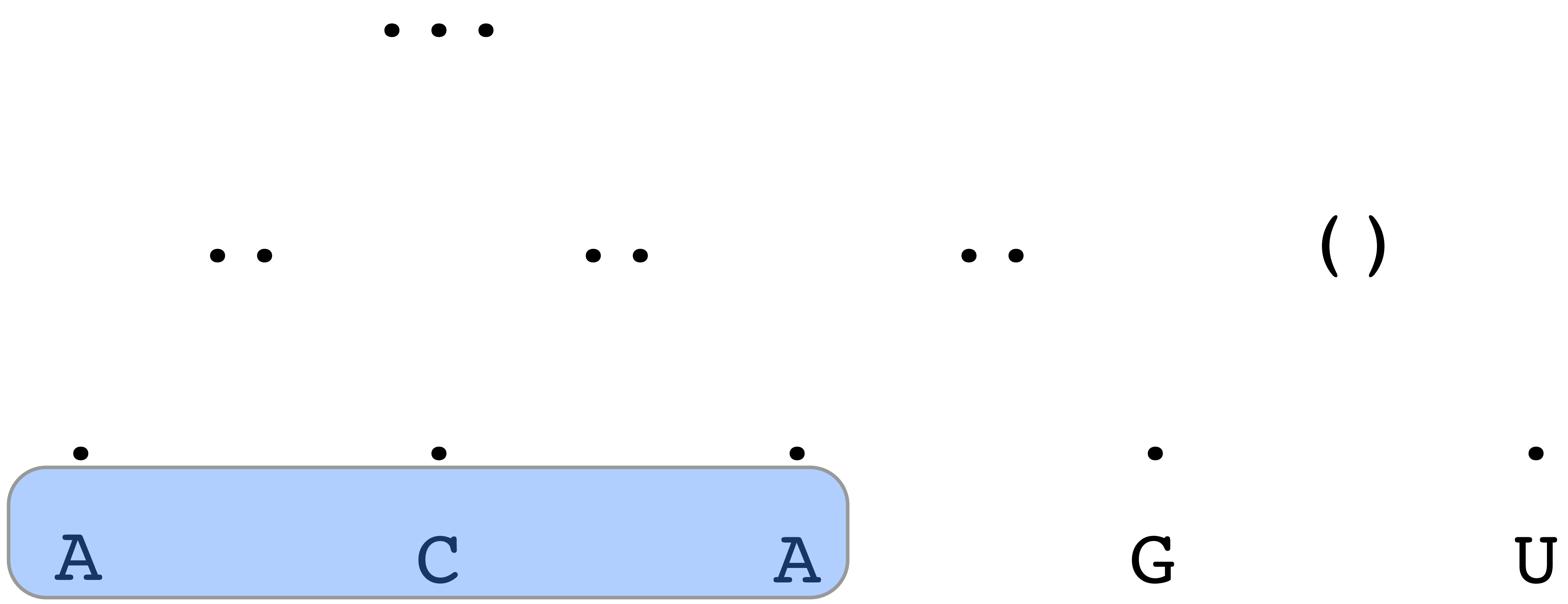

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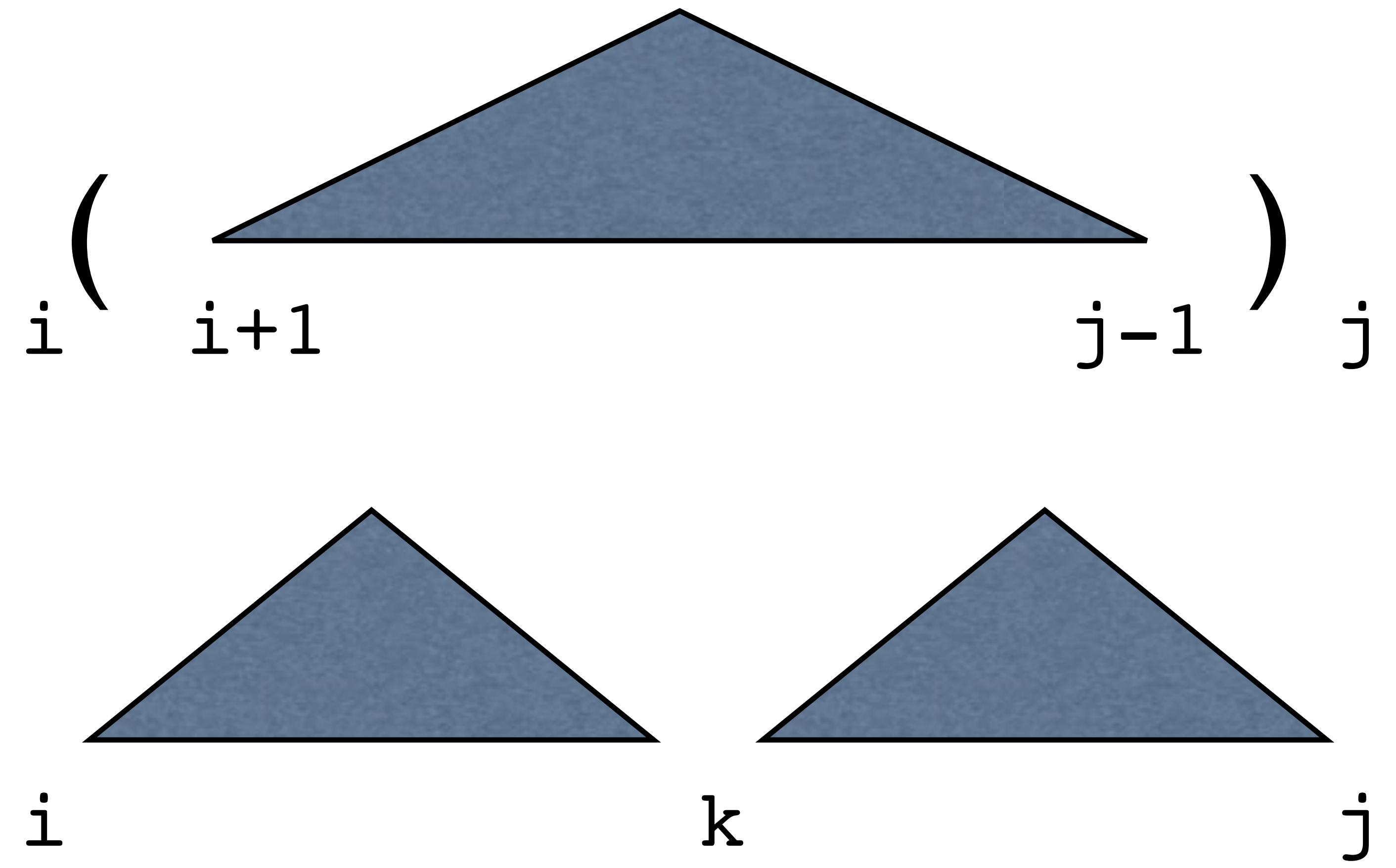

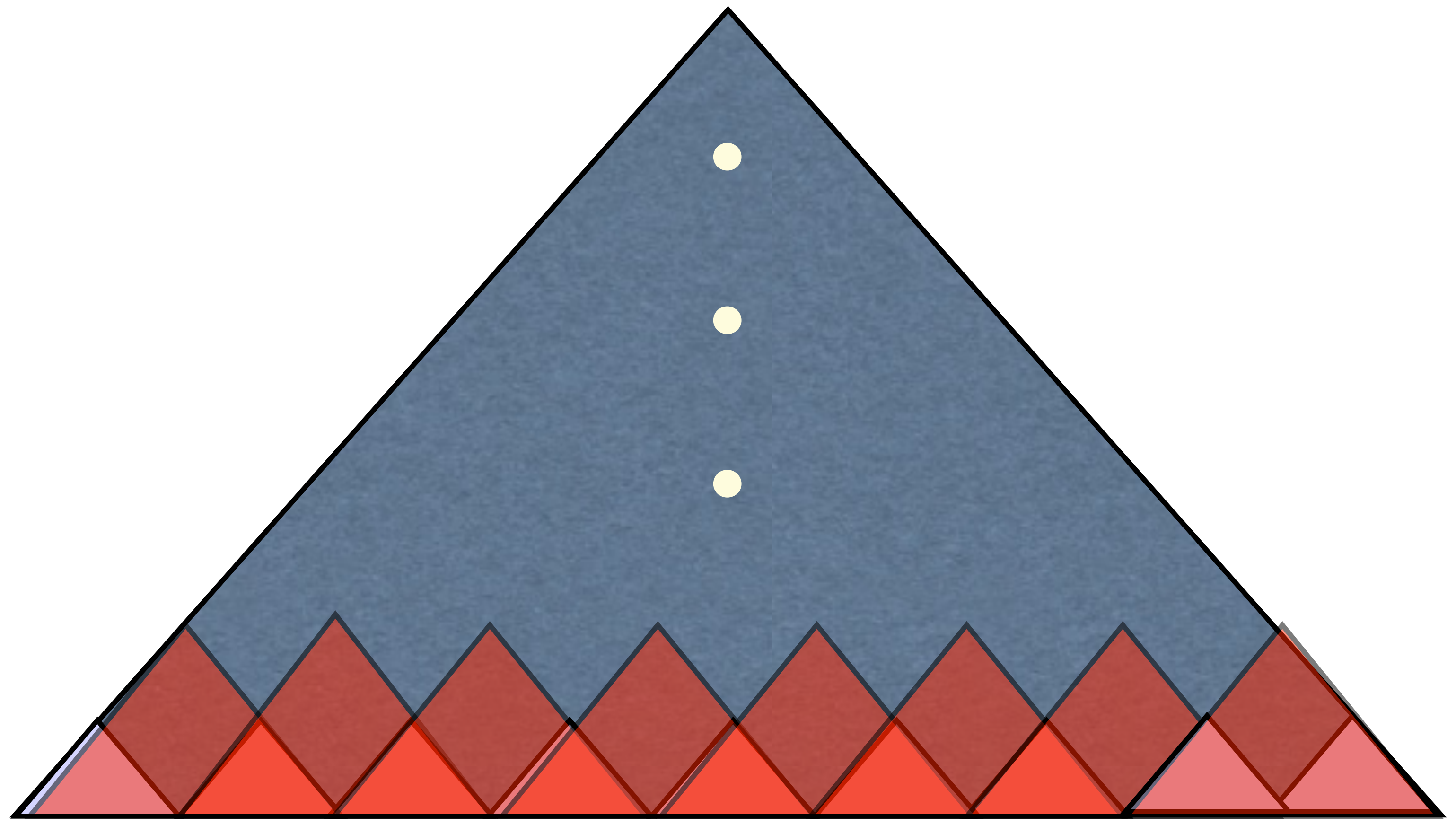

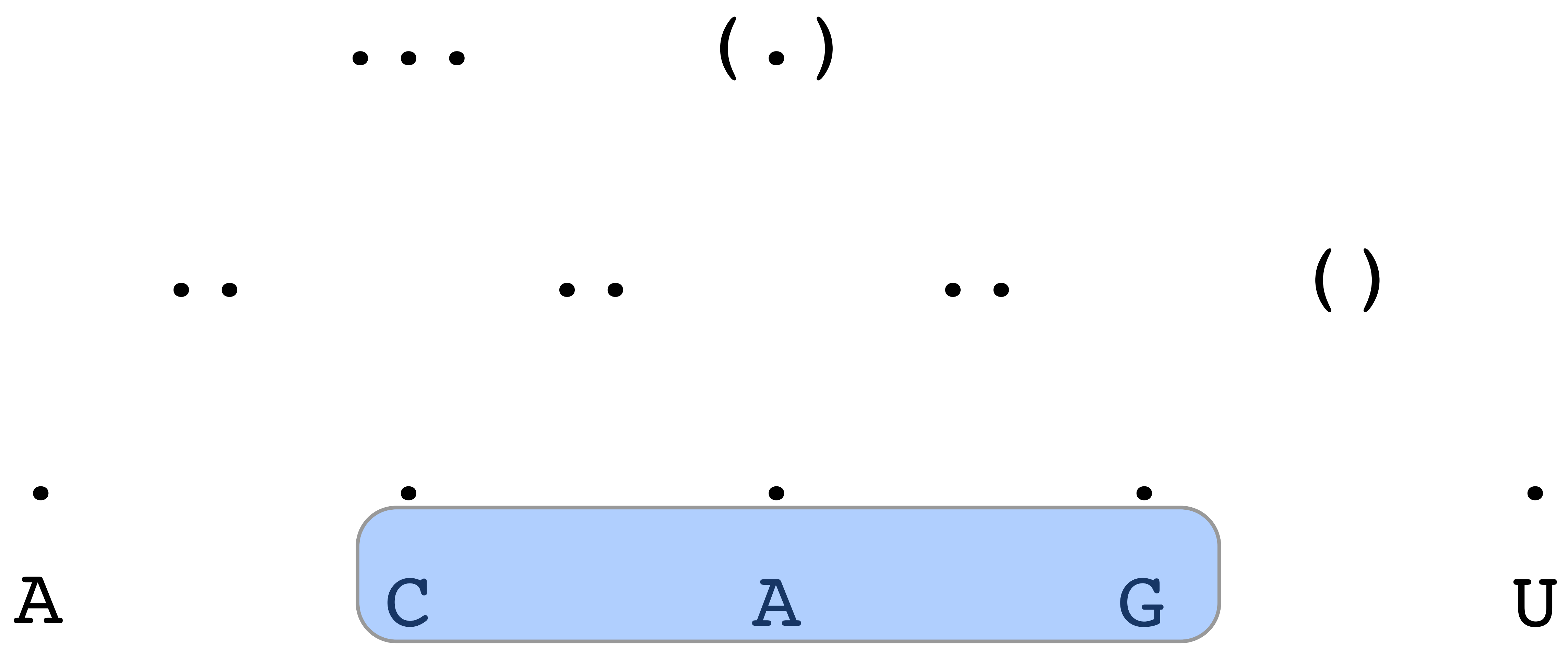

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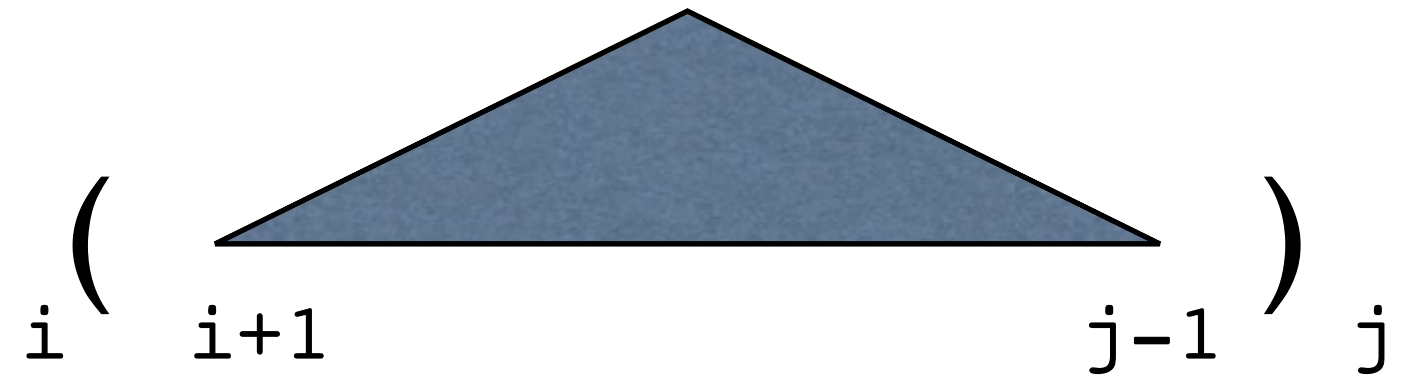

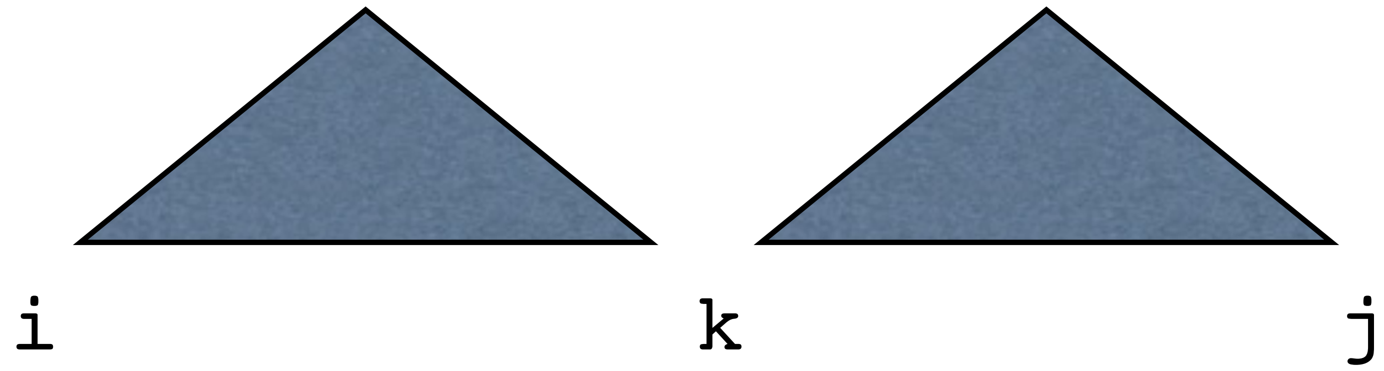

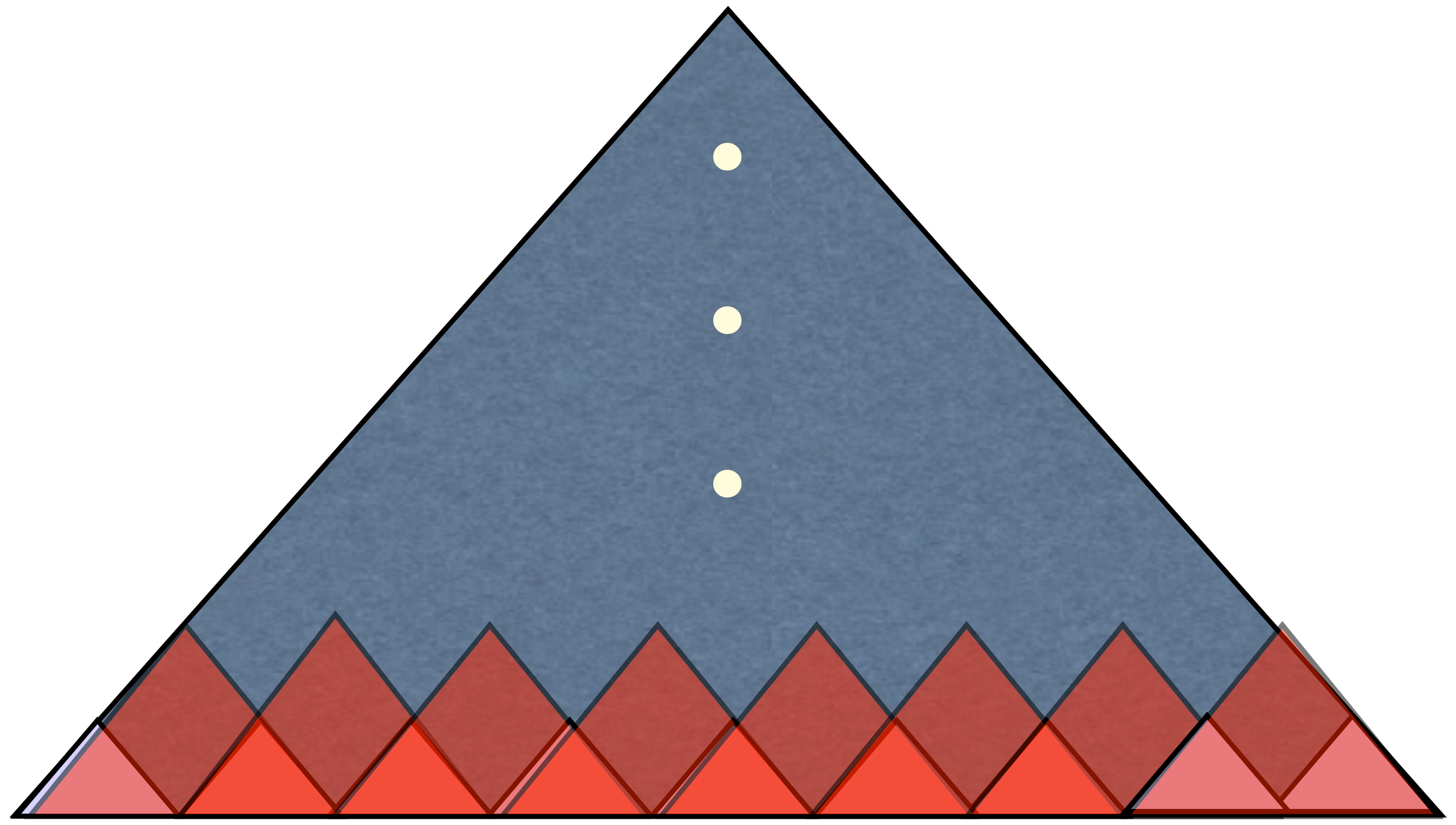

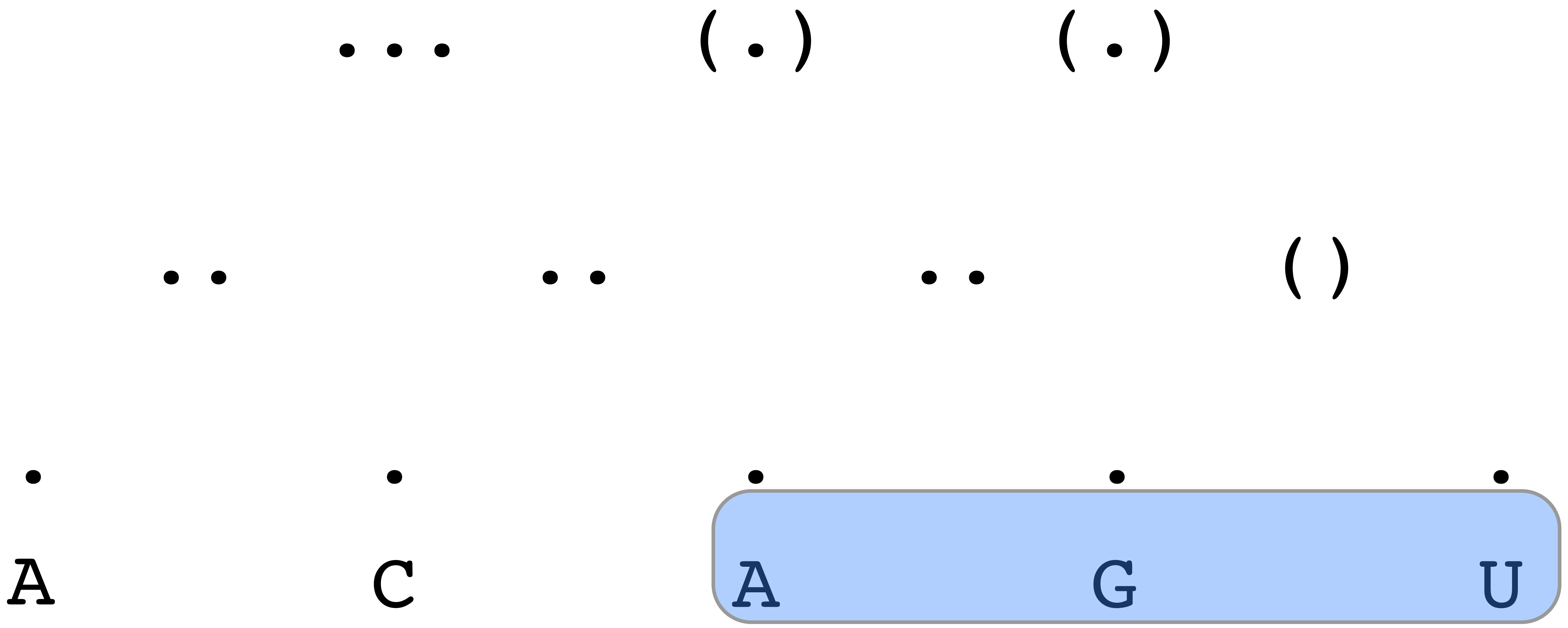

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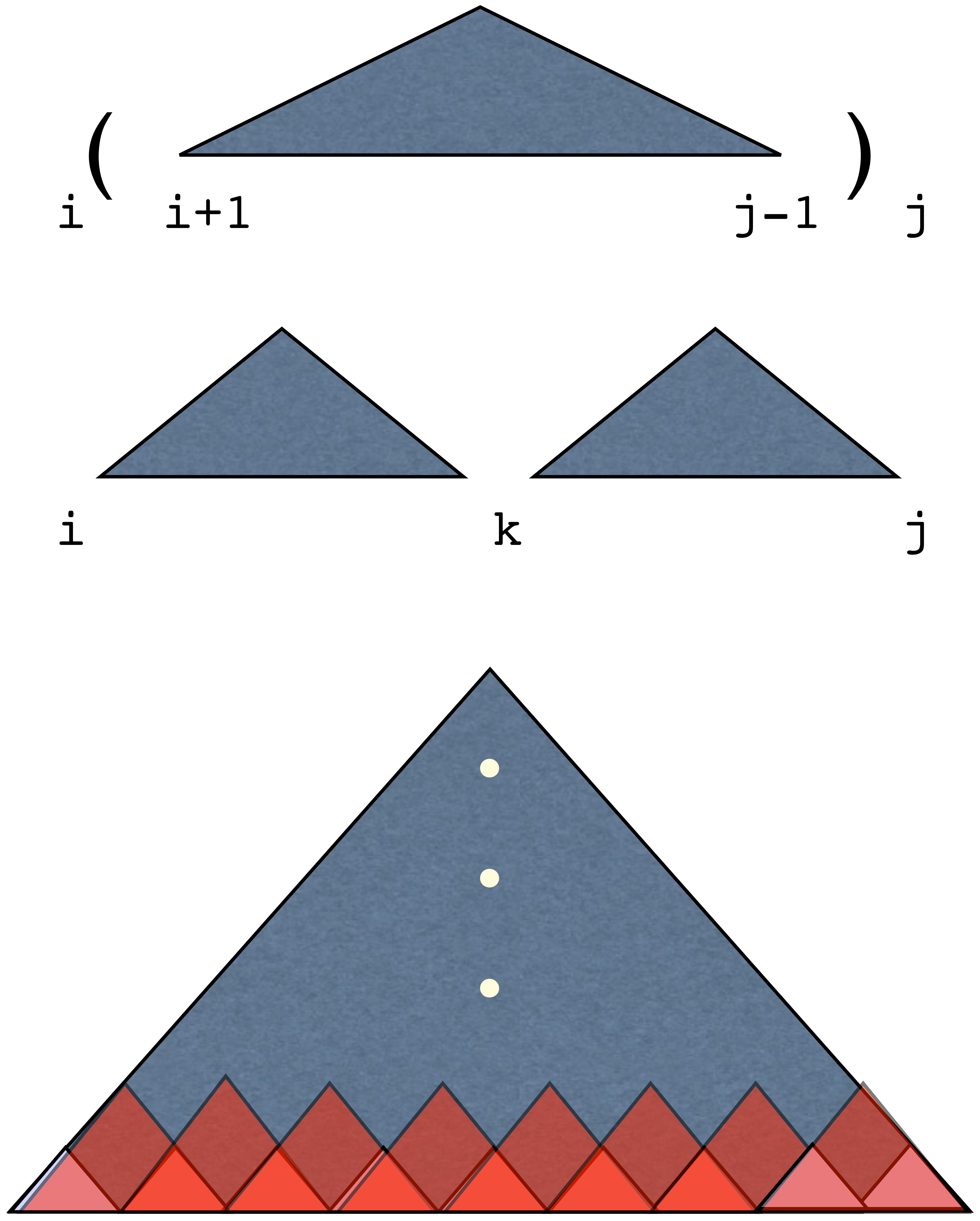

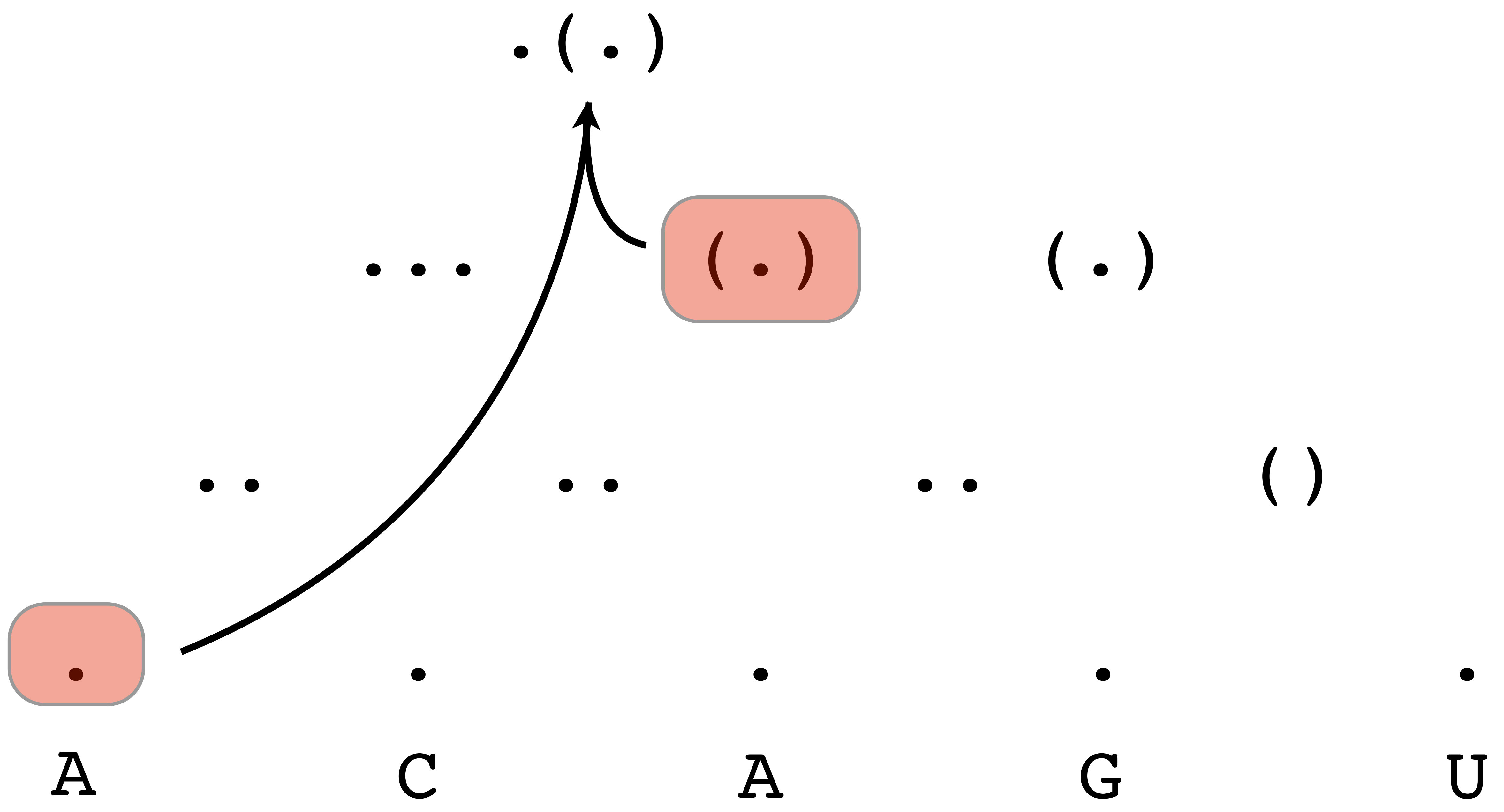

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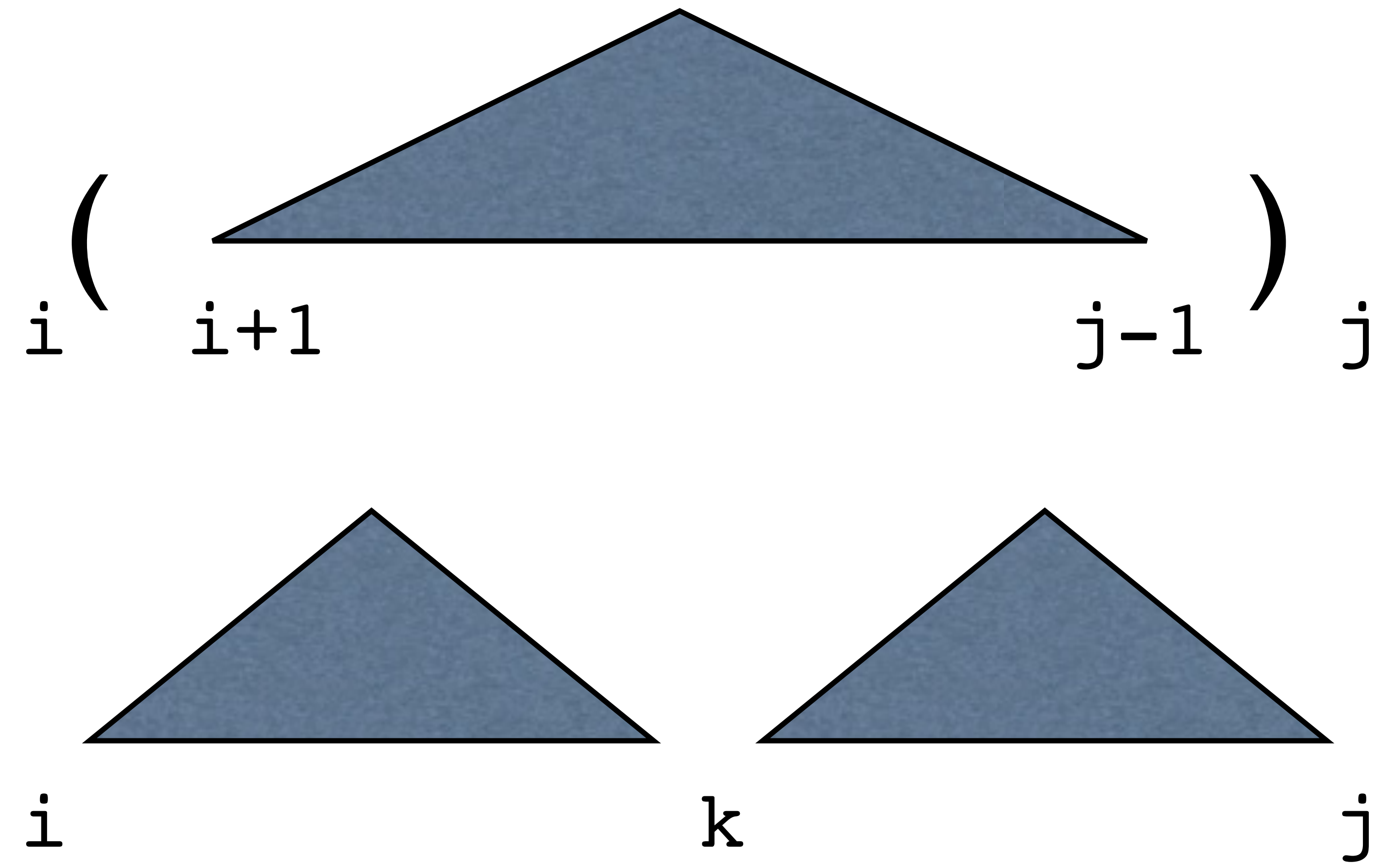

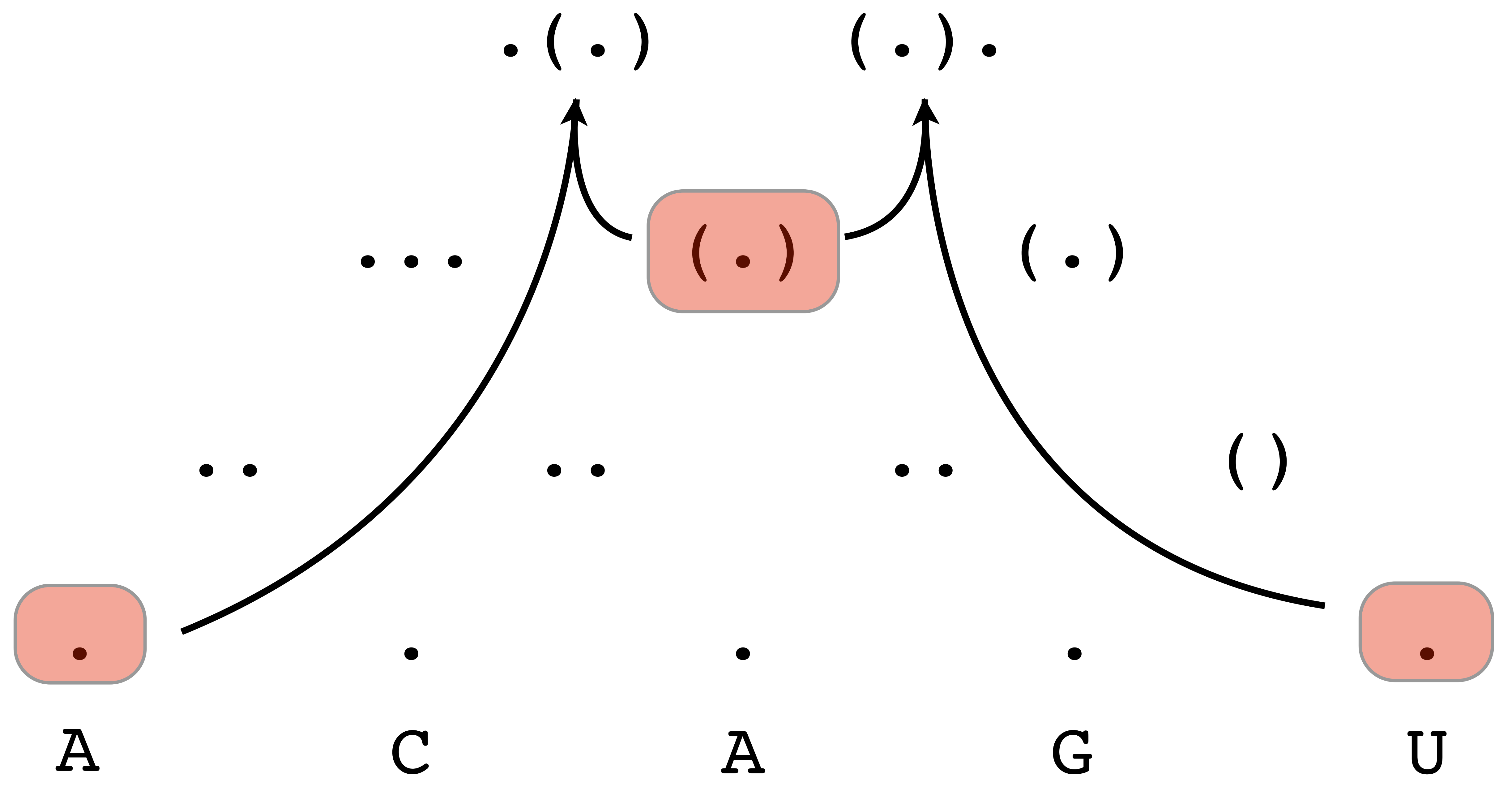

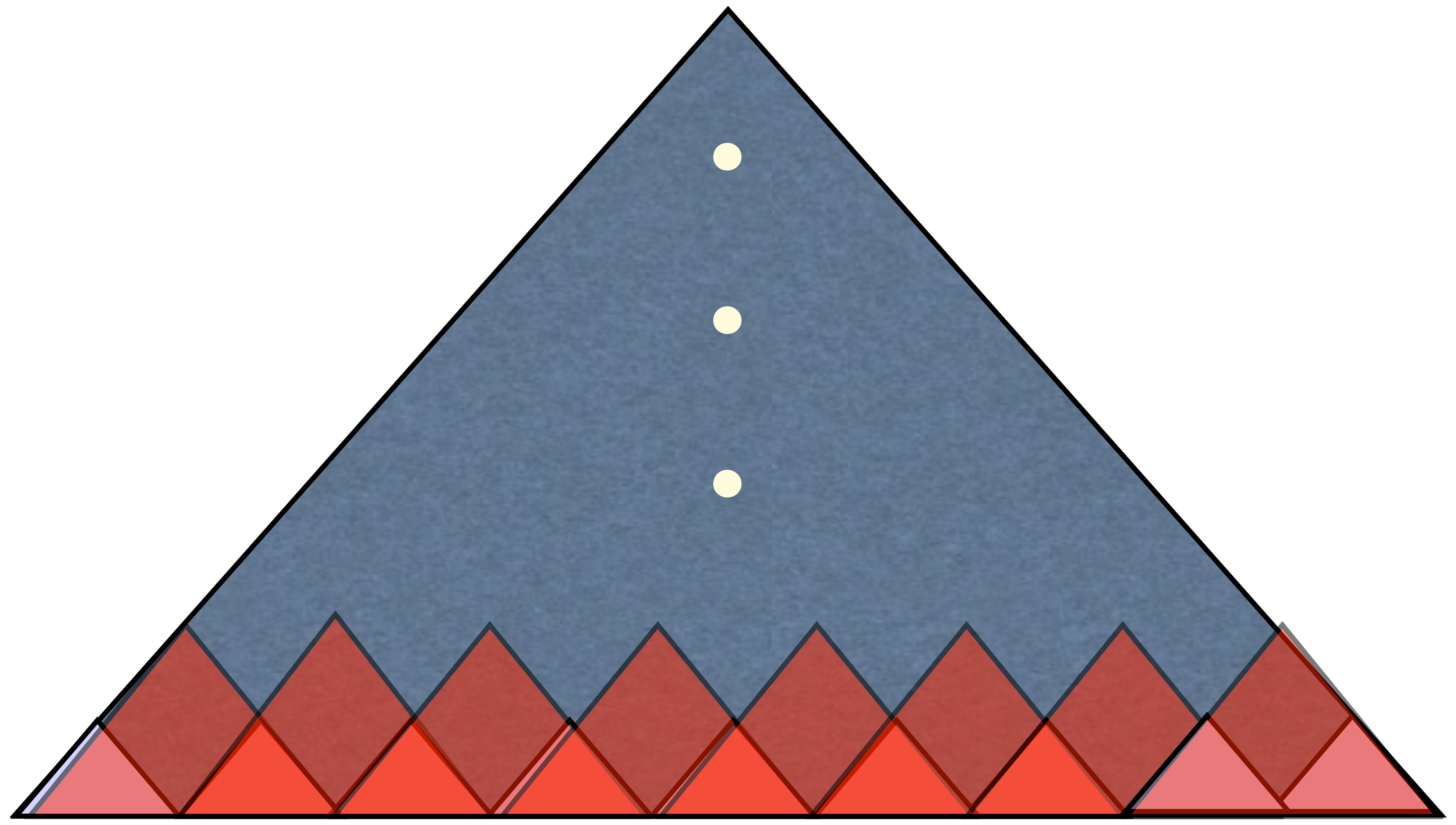

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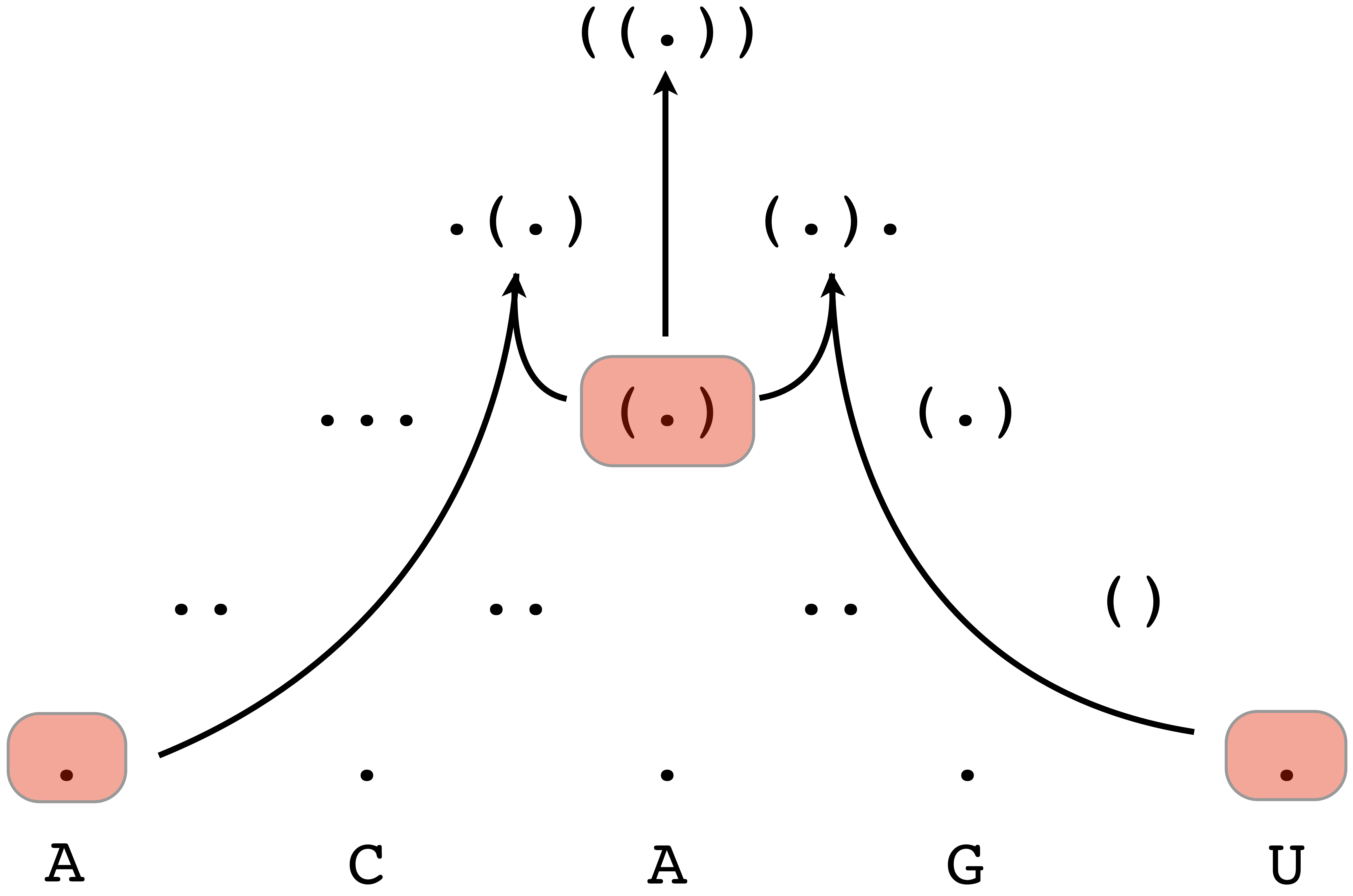
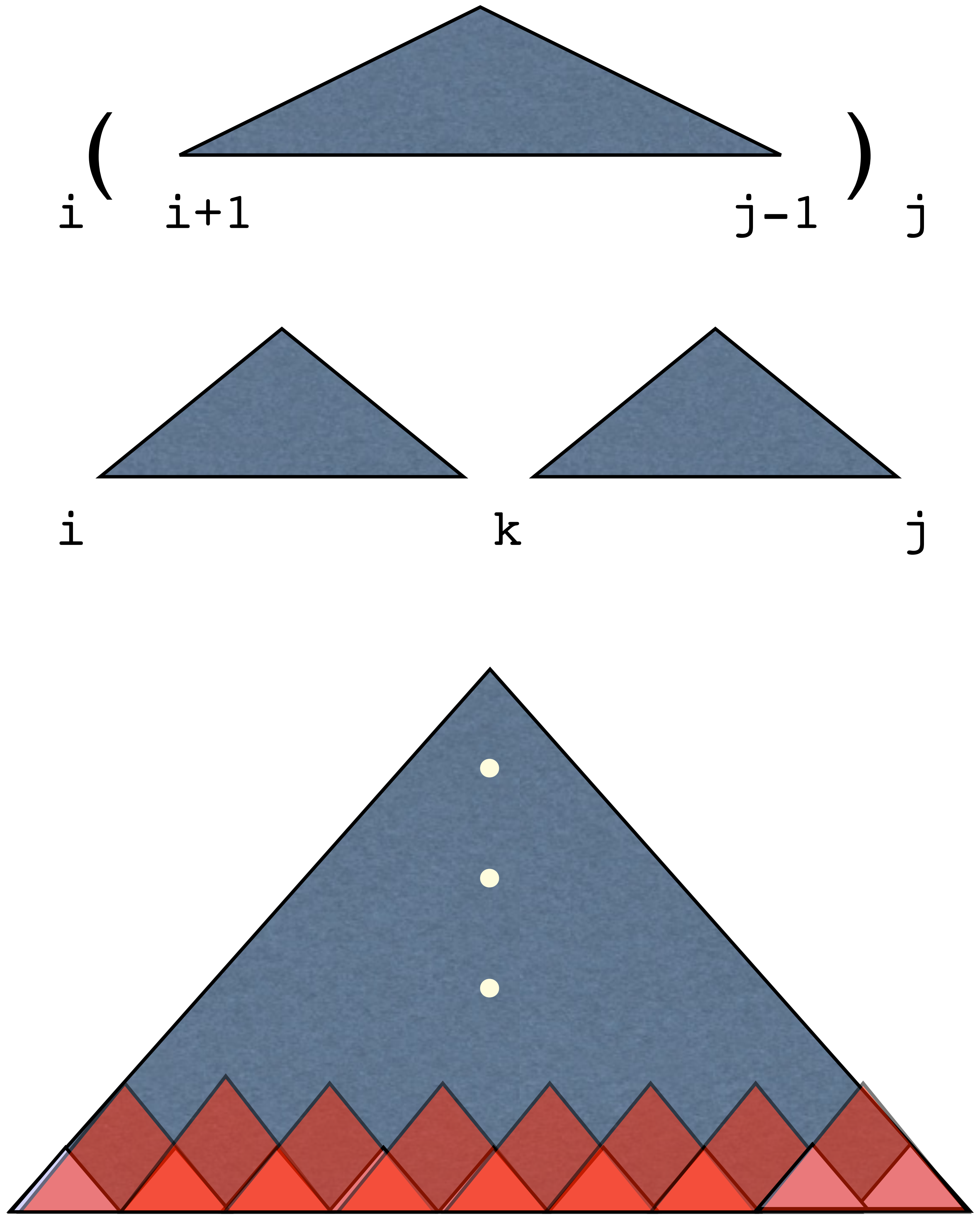
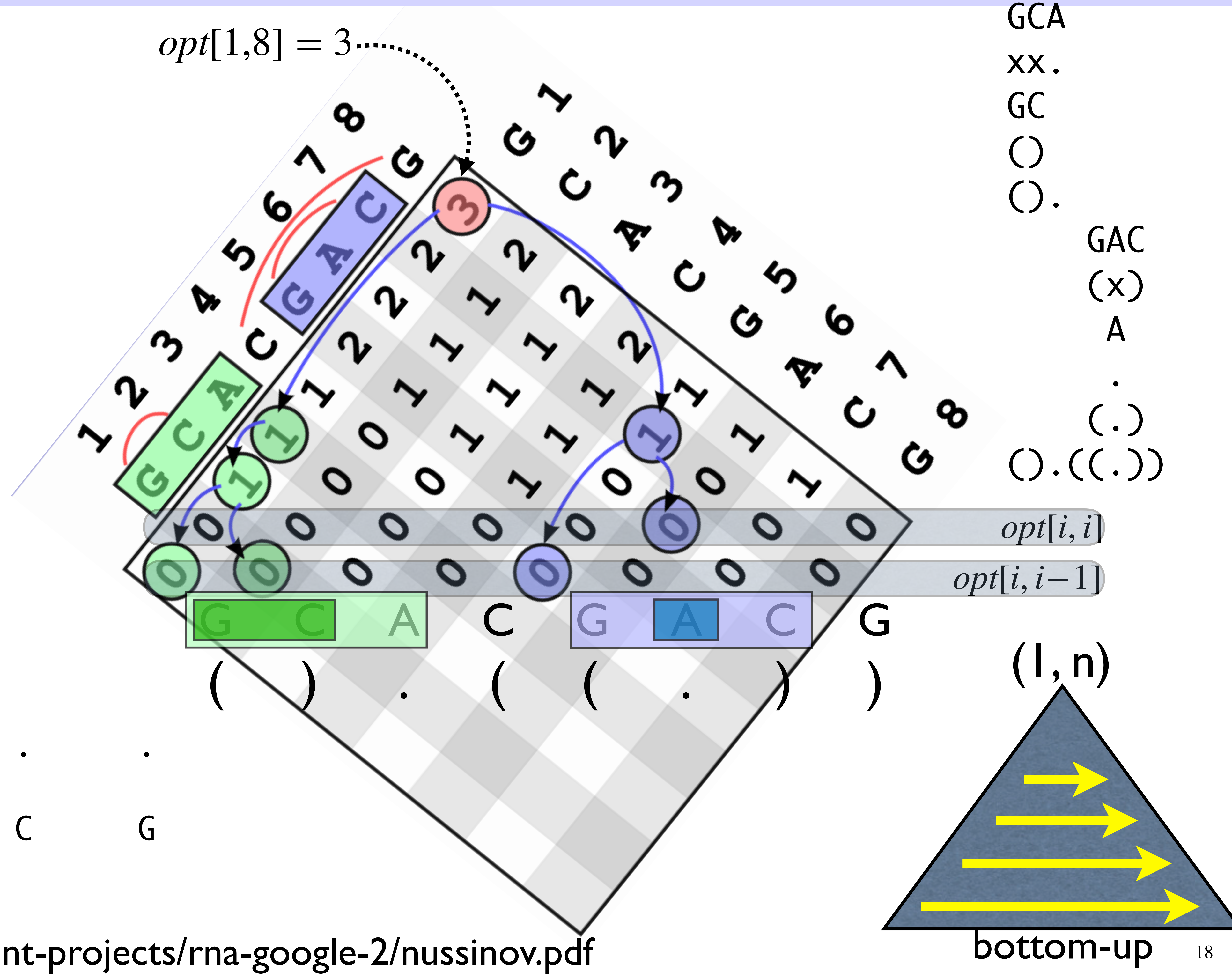




RNA Folding Example (1-best)

12345678
GCACGACG
xxx(xxx)
GCA
xx.
GC
()
().
GAC
(x)
A
.
(.)
().((.))



. . . .

G C A C G A C G

https://ad-publications.cs.uni-freiburg.de/student-projects/rna-google-2/nussinov.pdf

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```
12345678
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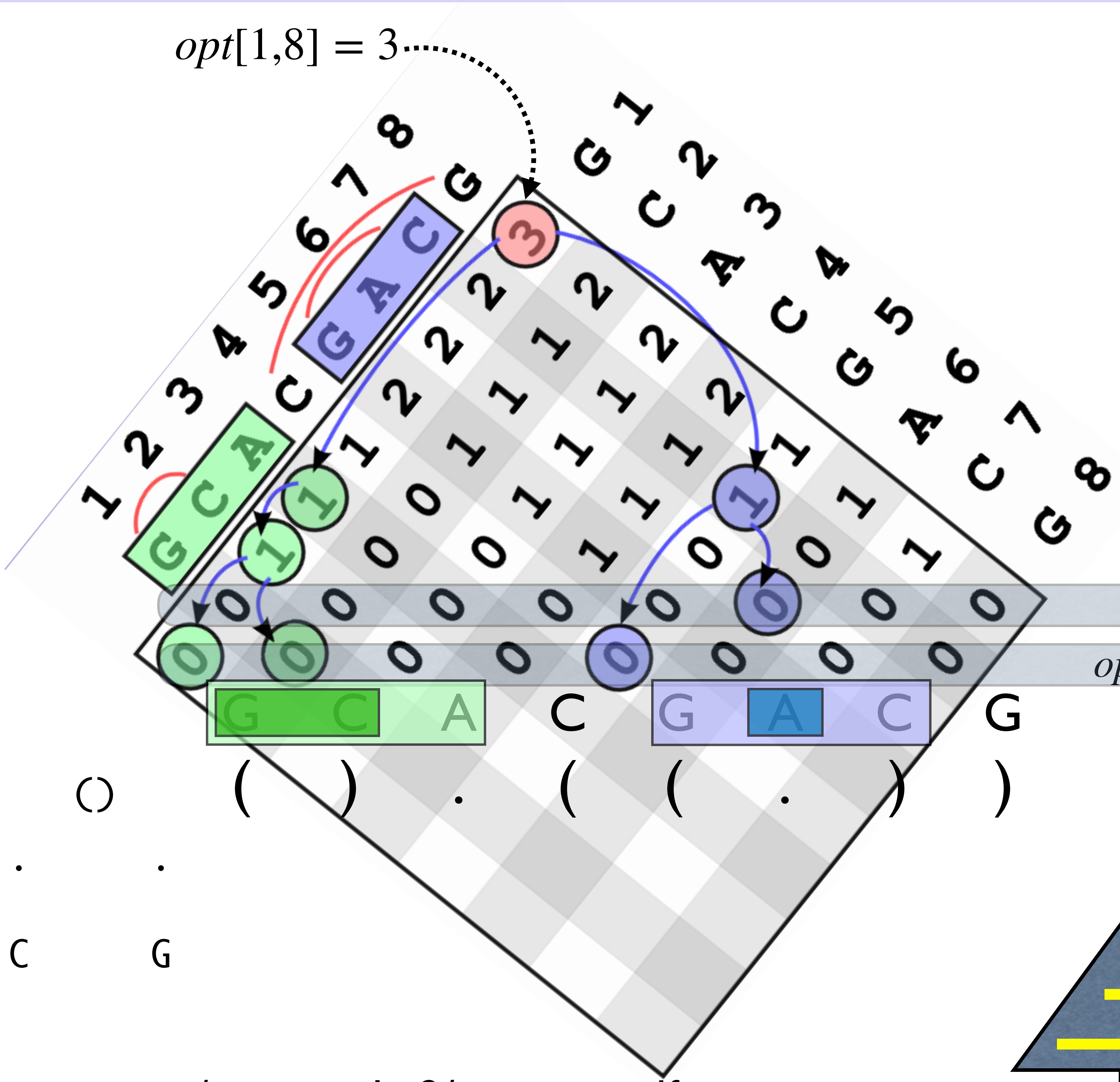
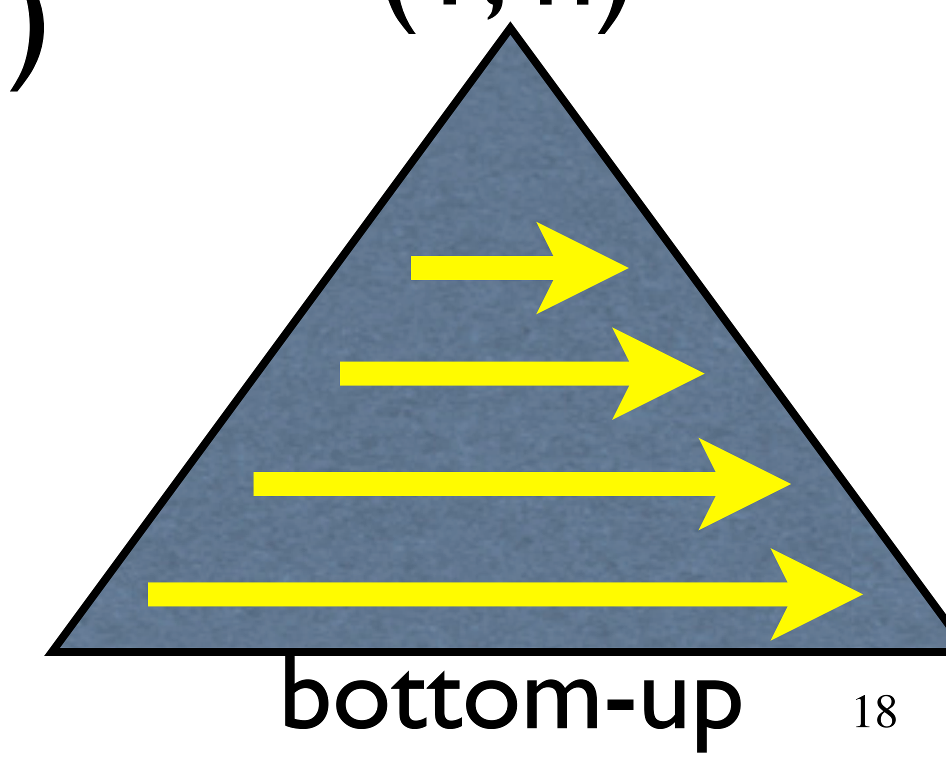
().
    GAC
    (x)
     A
     .
    (.)
().((.))
```



() .. .. () .. .. ()

. . . .

G C A C G A C G



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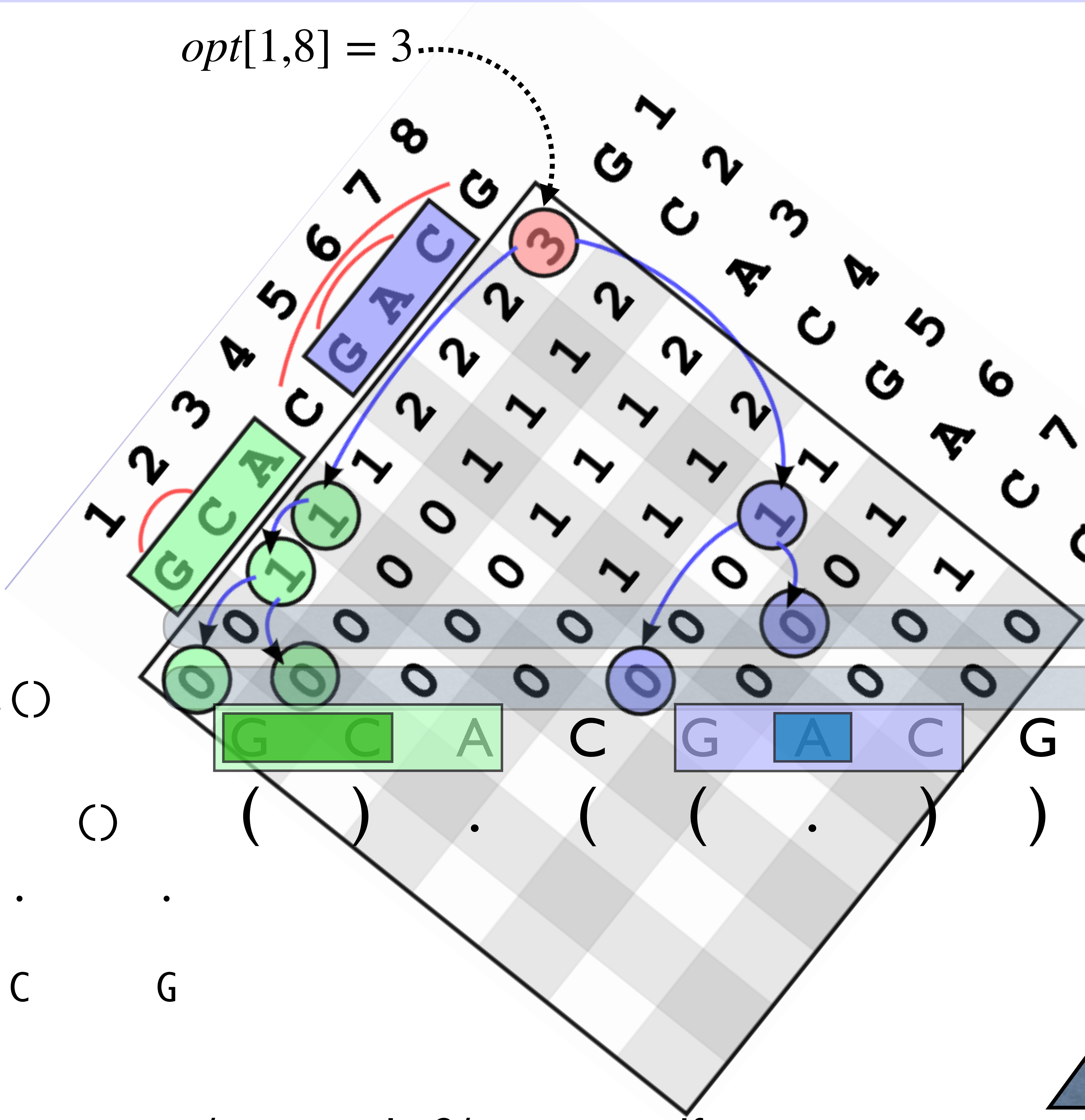


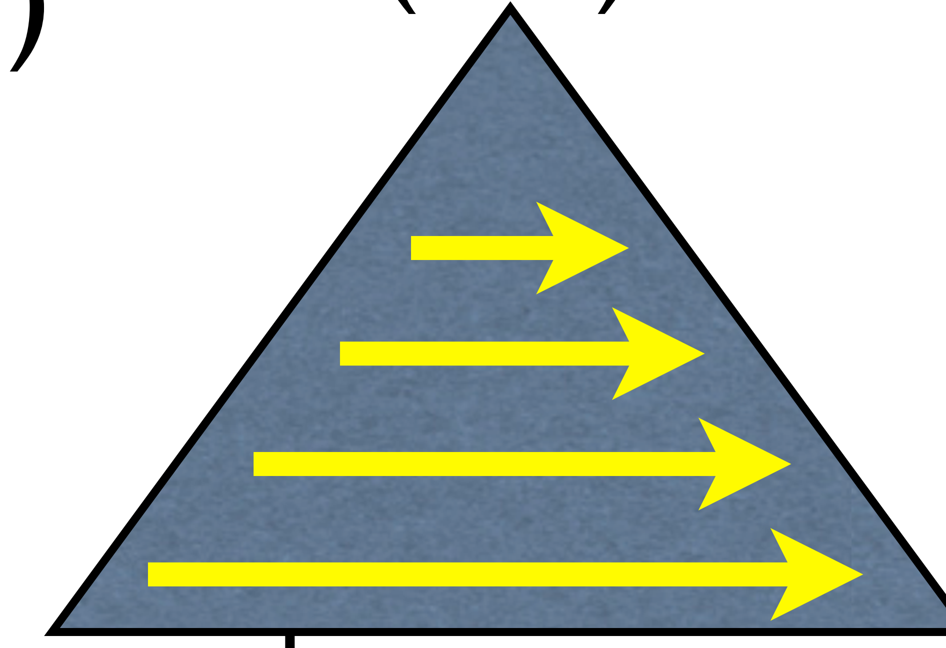

```
().    ...    .()    ().    (.)    .()

()    ..    ..    ()    ..    ..    ()

.    .    .    .    .    .    .    .

G    C    A    C    G    A    C    G
```

(1, n)



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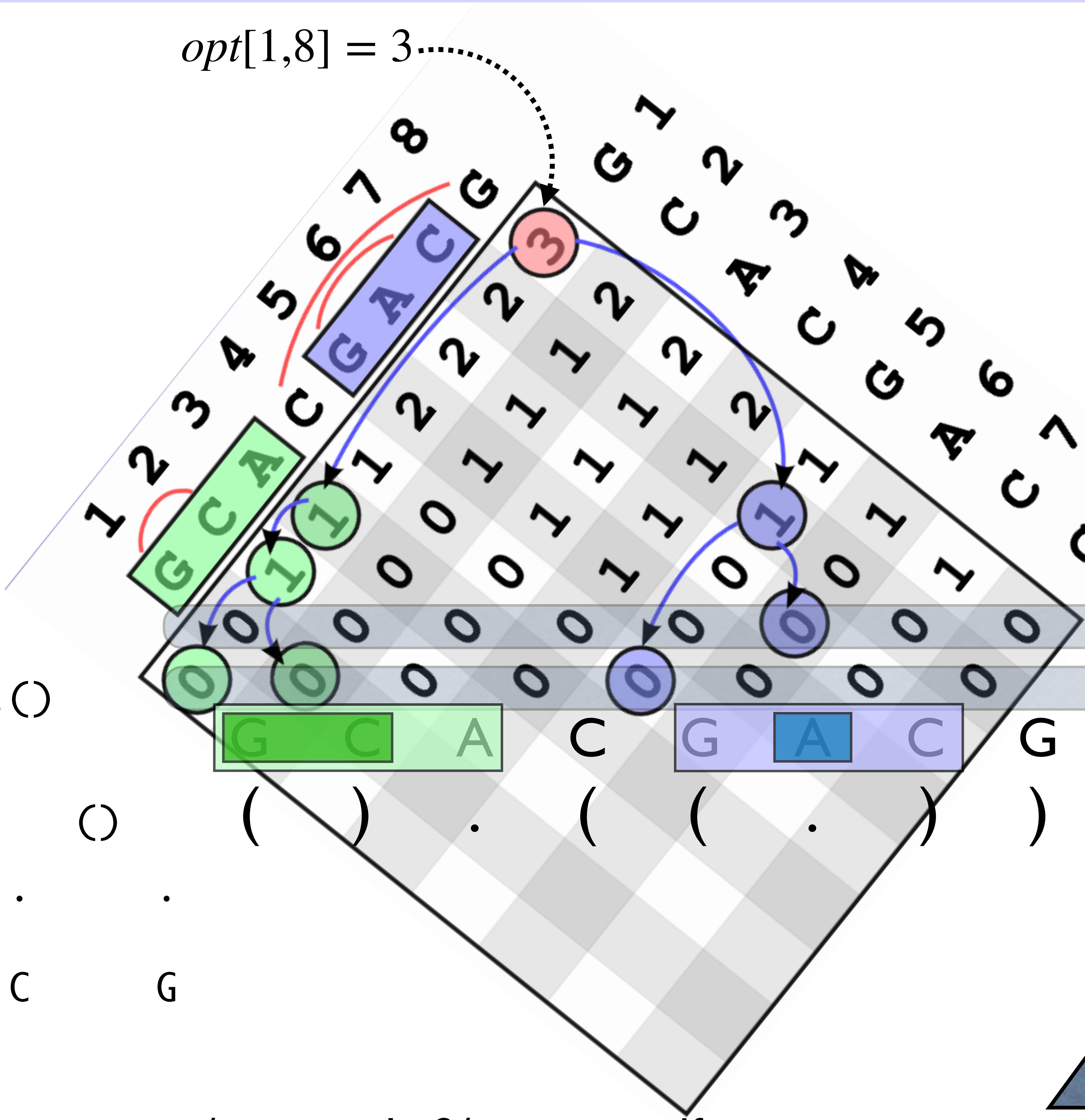
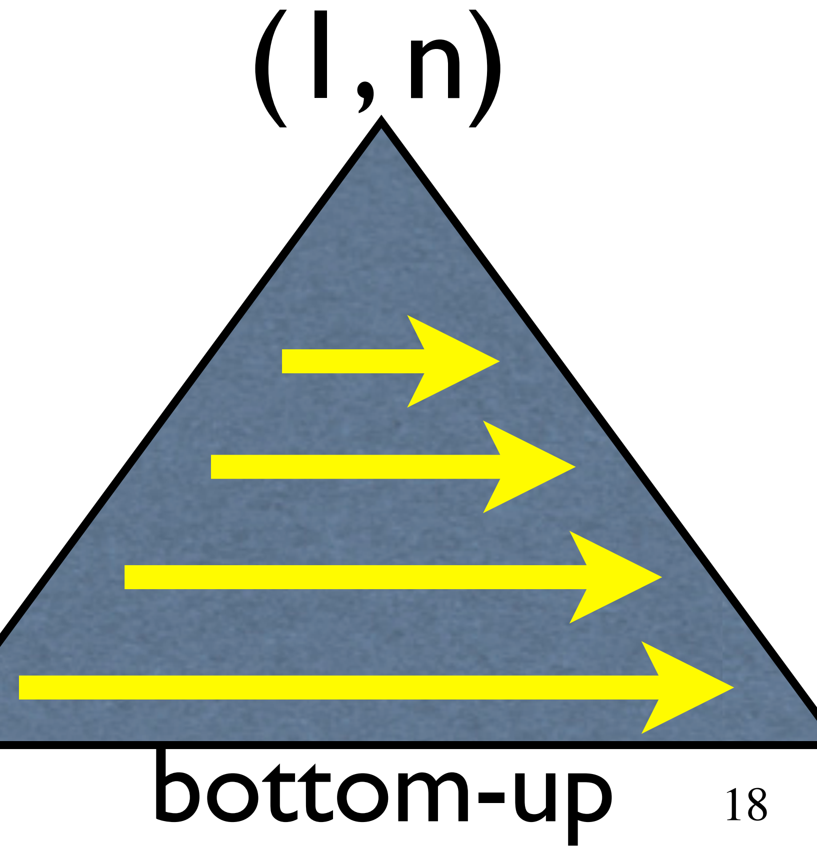
()
().
    GAC
    (x)
     A
     .
    (.)
().((.))
```

```
().. ..() .(). ().. ..()
(). ... .() (). (.) .()
() .. .. () .. .. ()
. . . . . . . .
G C A C G A C G
```





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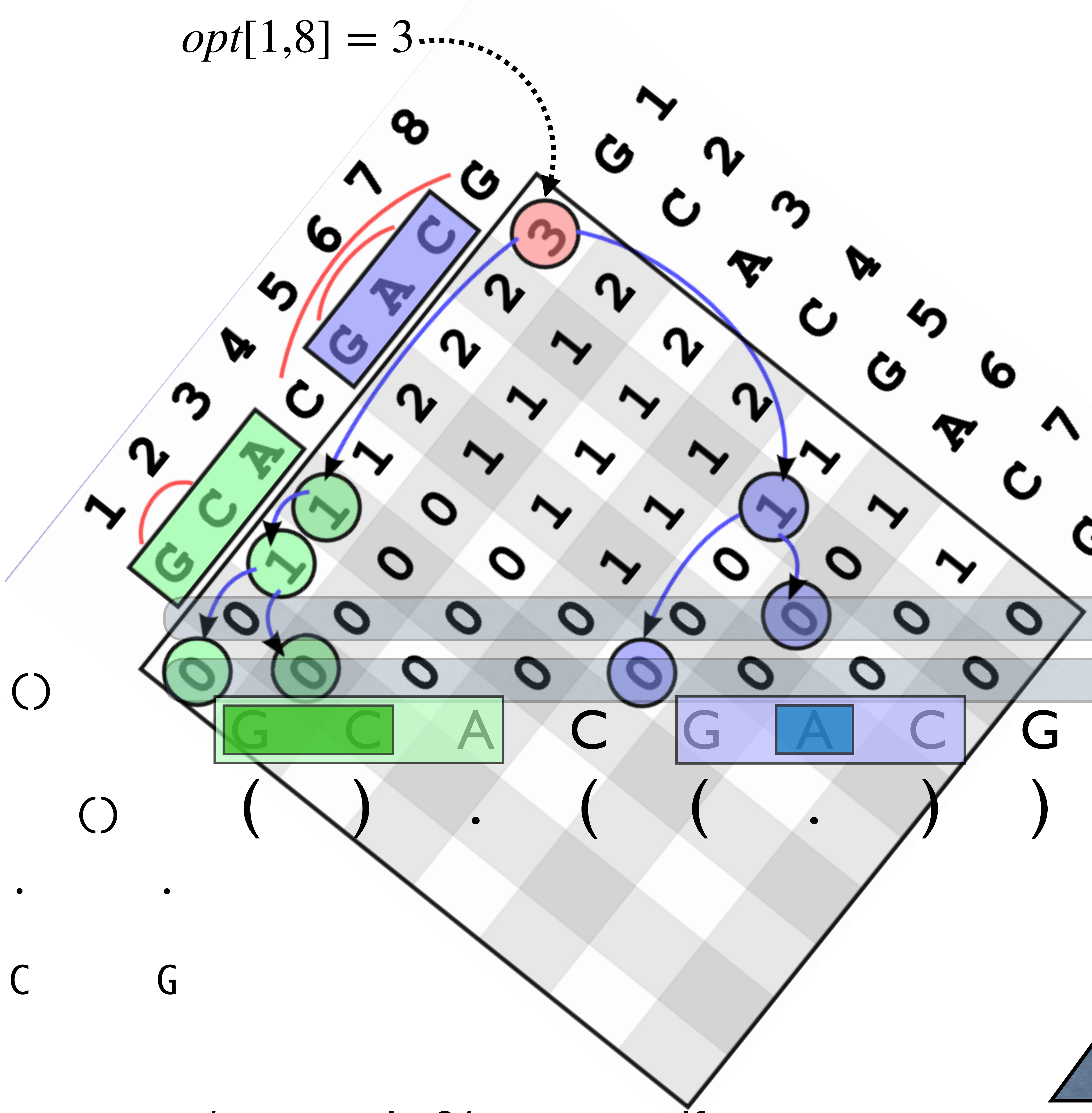
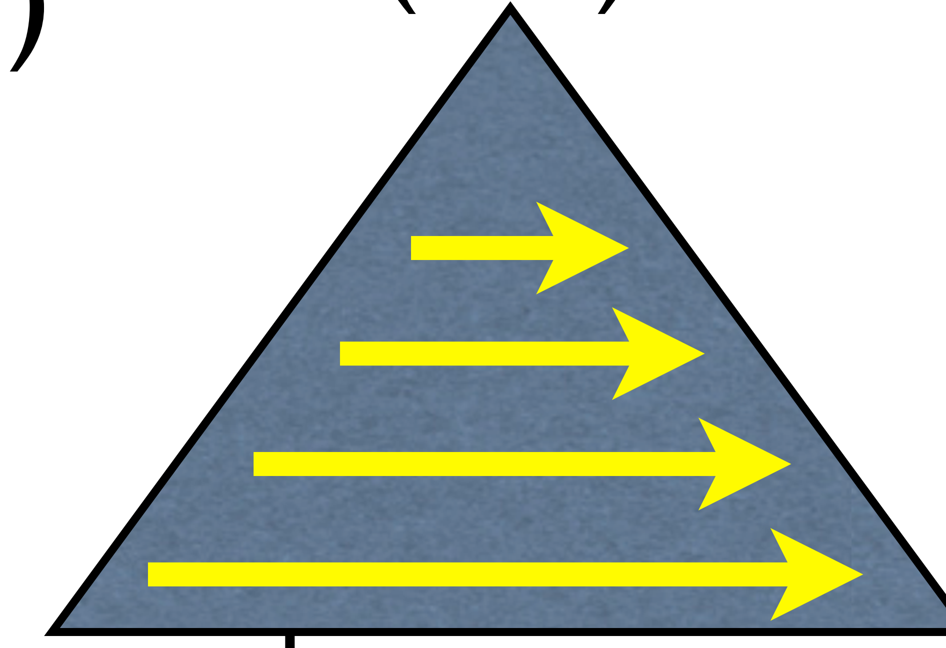
() .. .. () .. .. ()

. . . .

G C A C G A C G





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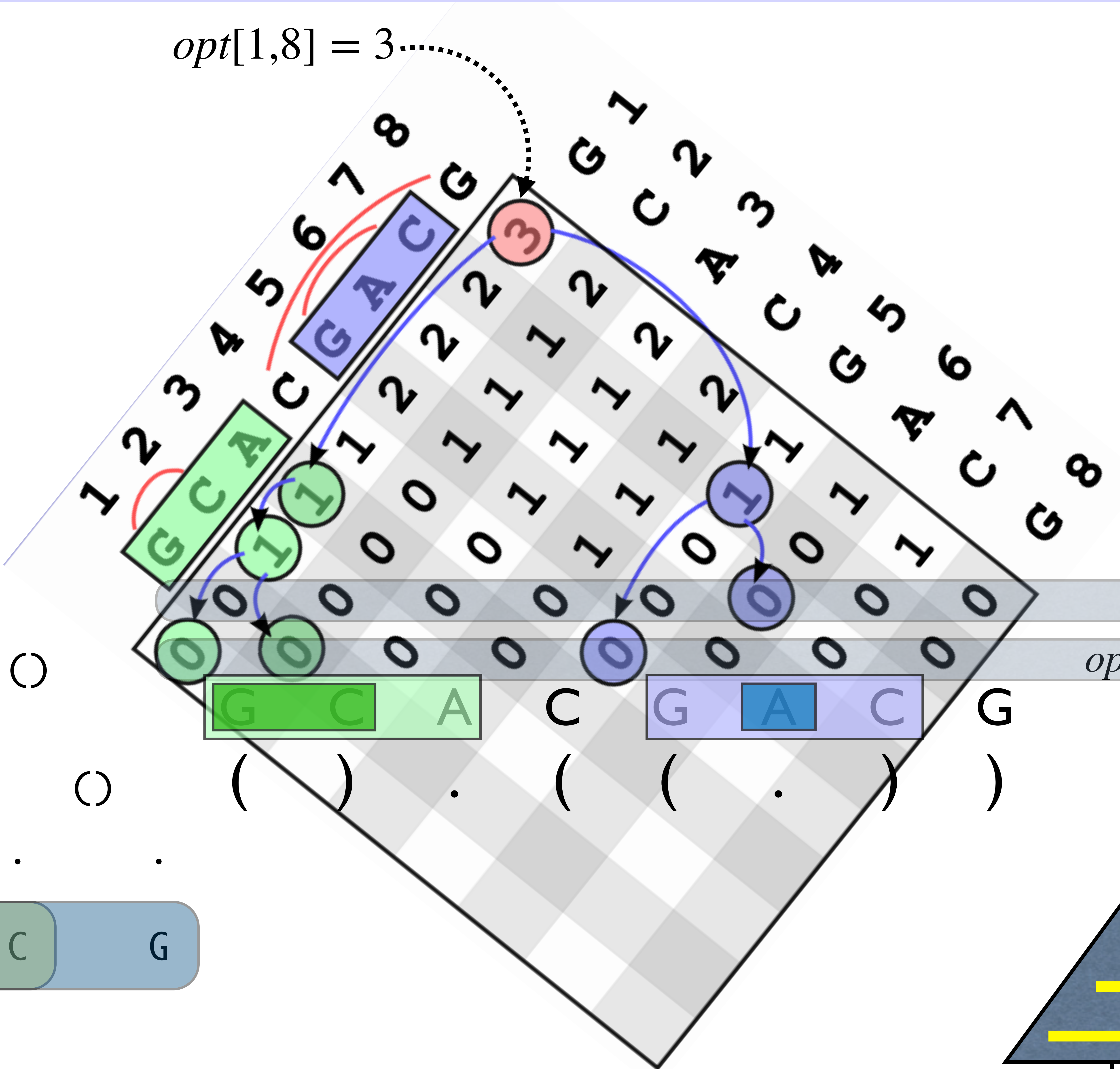
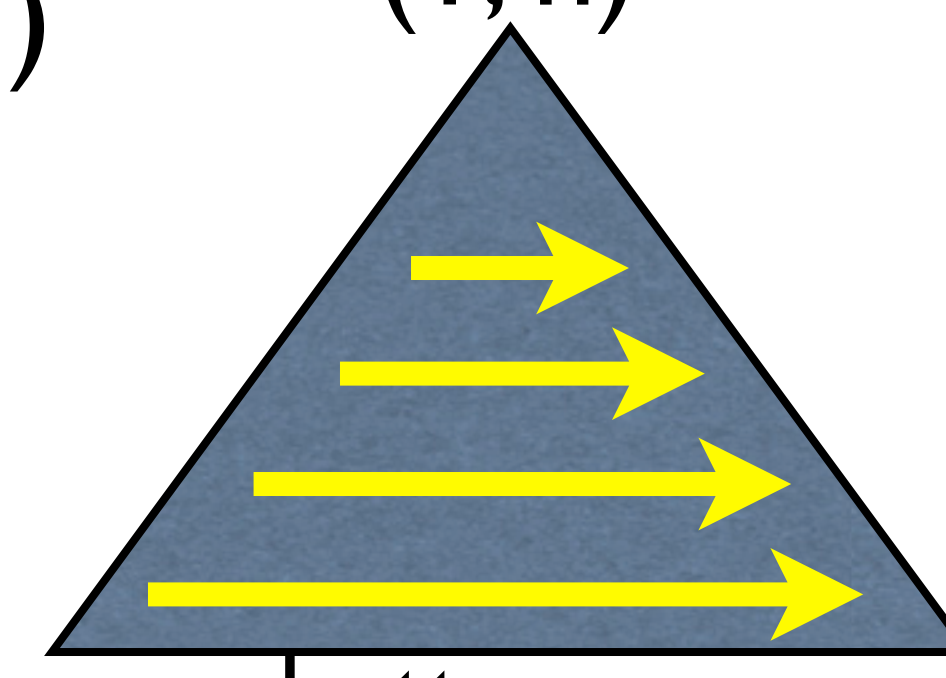
().((.))
```

```
().()..  ..().()
().(). ..(). .().()
().()  ..().  .()..  ().()
()..  ..()  .().  ()..  ..()
().  ...  .()  ().  (.)  .()
()  ..  ..  ()  ..  ..  ()
.  .  .  .  .  .  .  .
G  C  A  C  G  A  C  G
```





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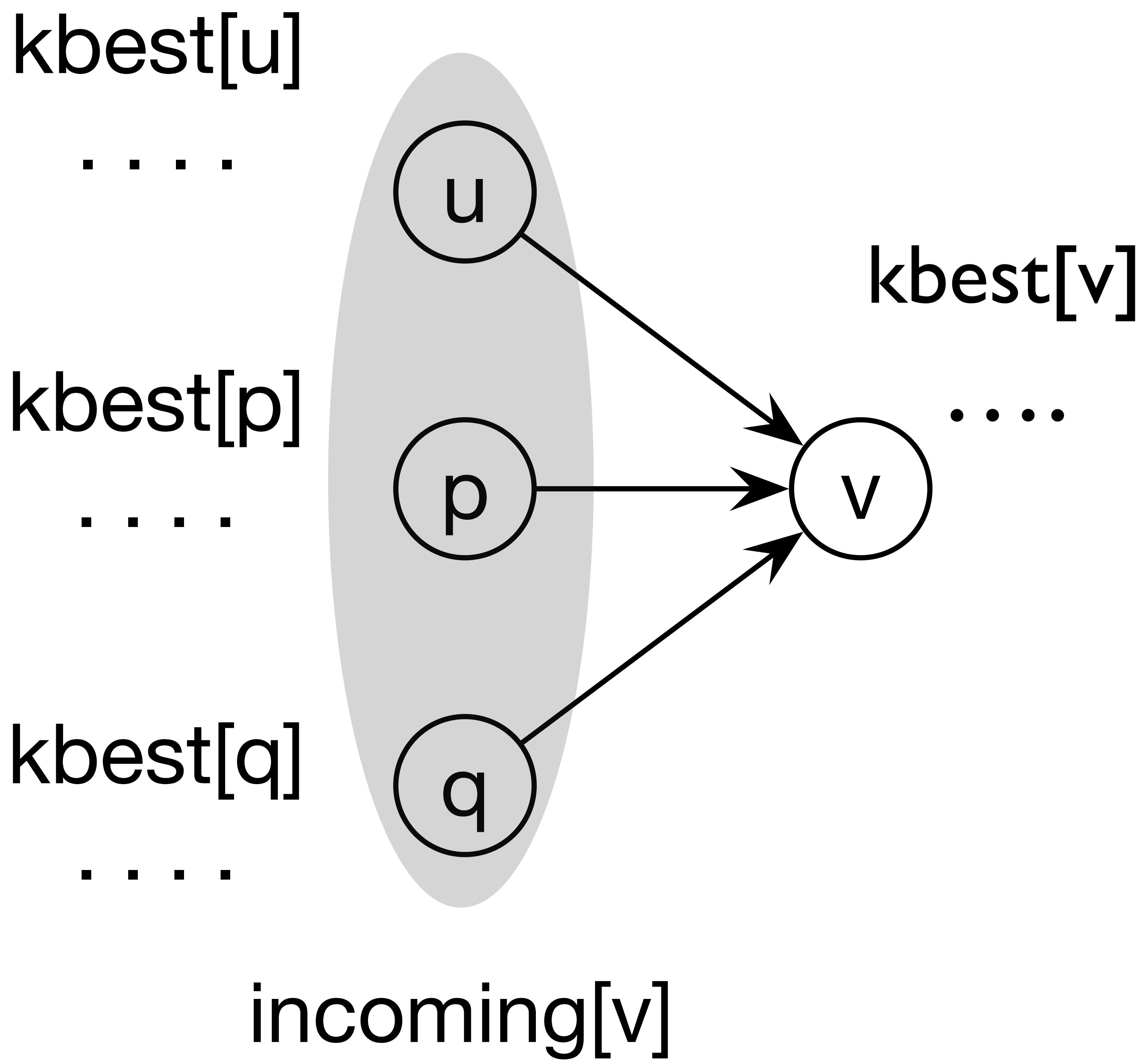
From 1-best to k-best

- each subproblem will now store top-k best answers instead of a single best
- we'll first extend Viterbi on DAGs to k-best Viterbi
- then extend generalized Viterbi on DAHs (e.g., CKY or Nussinov) to k-best

k-best Viterbi on Graph

- simple extension of Viterbi to solve k-best on graphs and hyper graphs

cf. teams problem in HW4



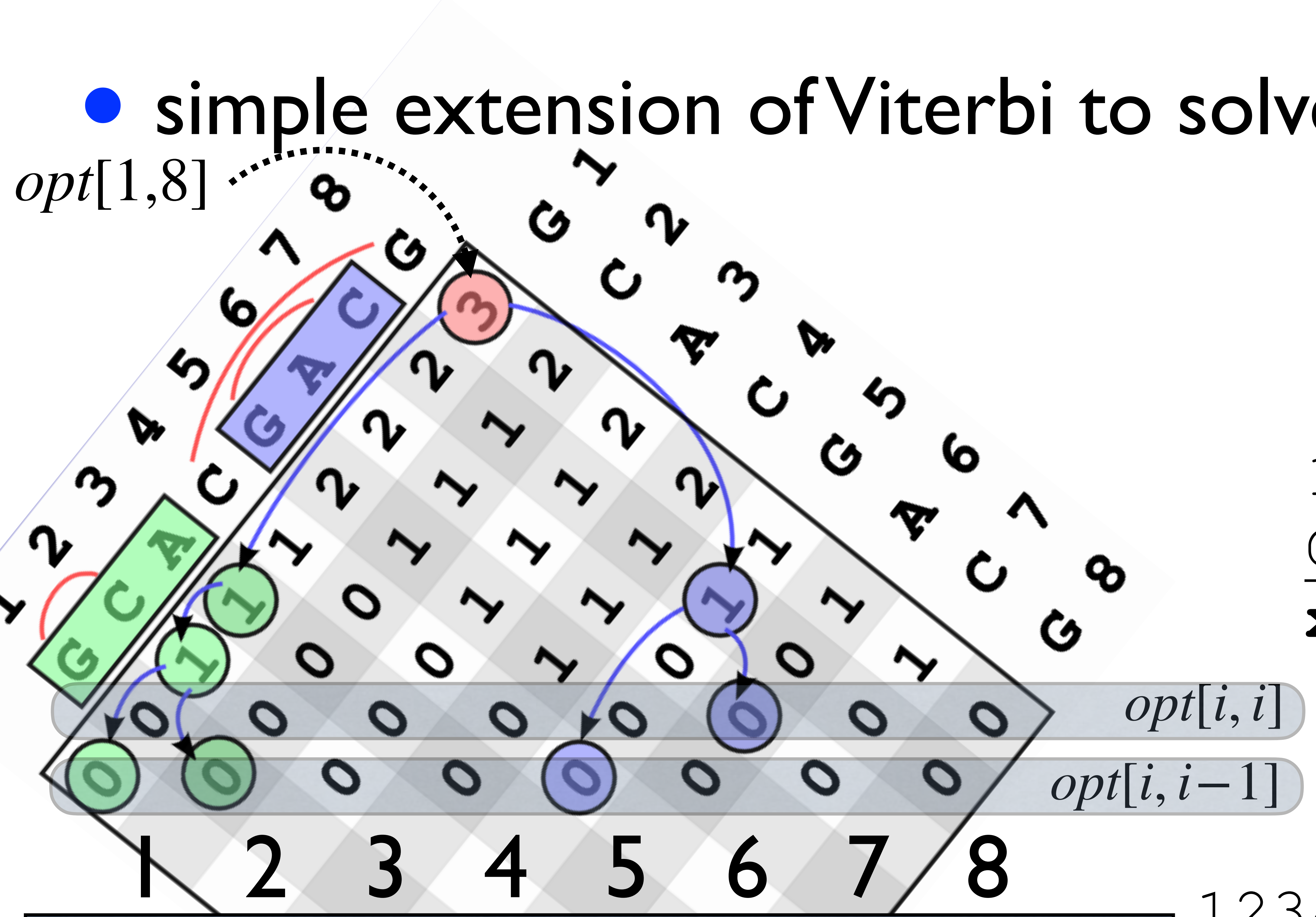
for each node v,
compute its kbest distances
from the kbest of each incoming node u

1-best: $O(E+V)$
k-best: $O(E + Vk \log d_{\max})$ where $d_{\max}$ is the max in-degree

can improve it to: (cf. midterm & teams, w/ quickselect)
k-best: $O(E + Vk \log k)$ (assume $k \ll d_{\max}$)
("most states do not have anybody on team USA")

k-best Viterbi on Hypergraph

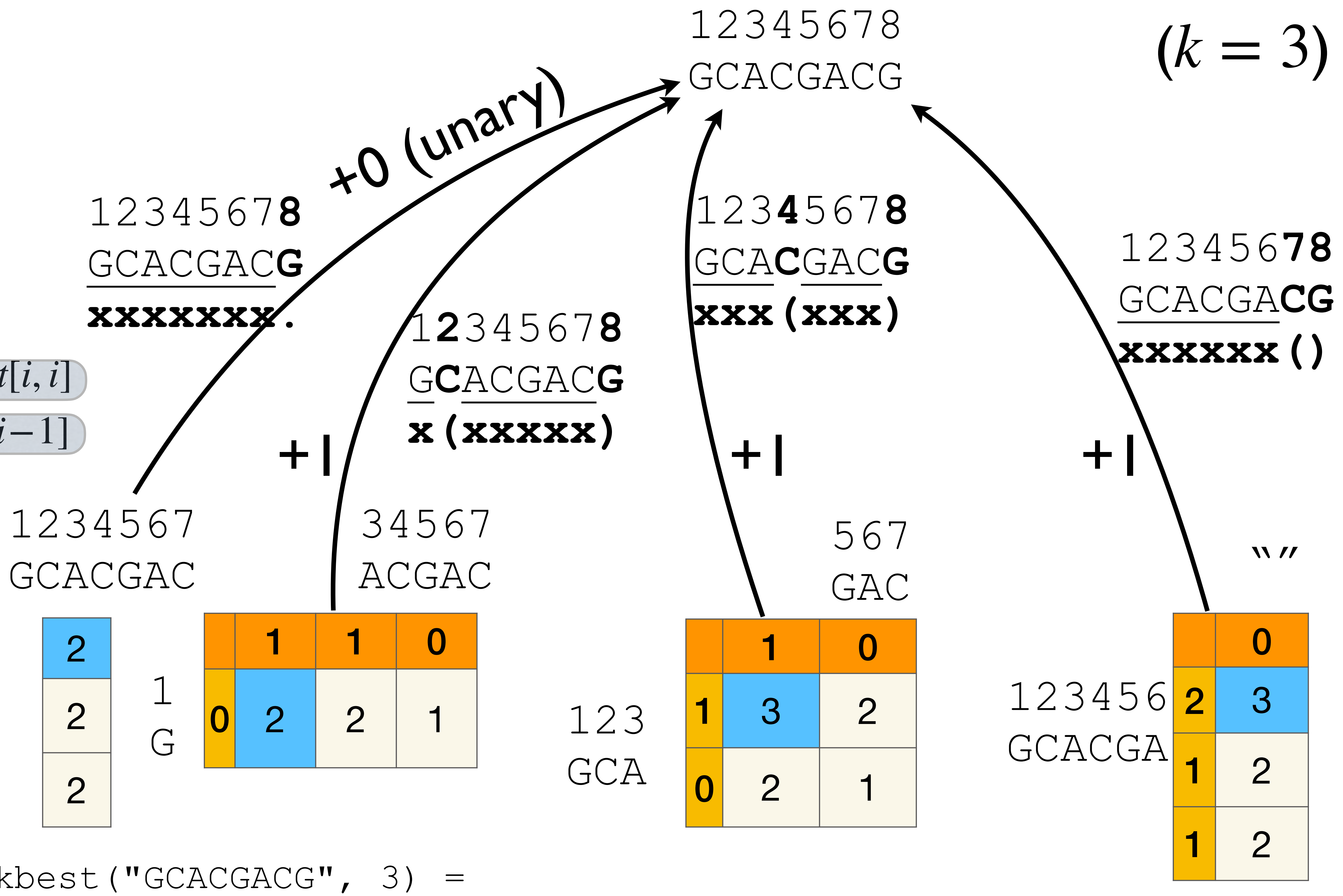
- simple extension of Viterbi to solve k-best on graphs and hyper graphs   cf. midterm



$$opt[i,j] = \bigoplus \begin{cases} opt[i,j-1], \\ \bigoplus_{i \le p < j} (opt[i,p-1] \otimes opt[p+1,j-1] \otimes 1) \end{cases}$$

$$opt[i,i] = opt[i,i-1] = 1_{\otimes}$$

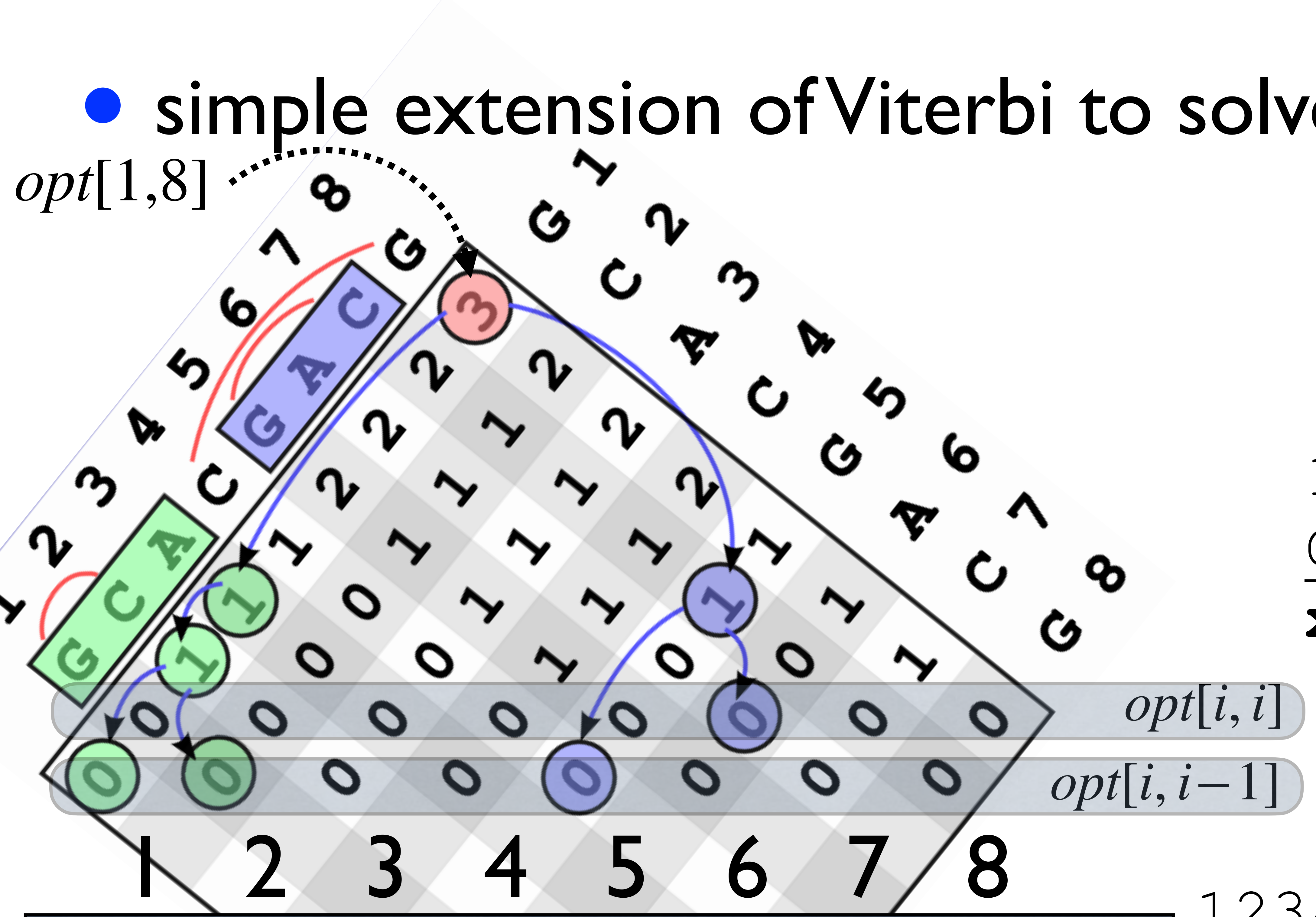
opt	⊕	⊗	$1_{\otimes}$
best	max	+	0
total	+	x	1



kbest("GCACGACG", 3) =
[]

k-best Viterbi on Hypergraph

- simple extension of Viterbi to solve k-best on graphs and hyper graphs   cf. midterm



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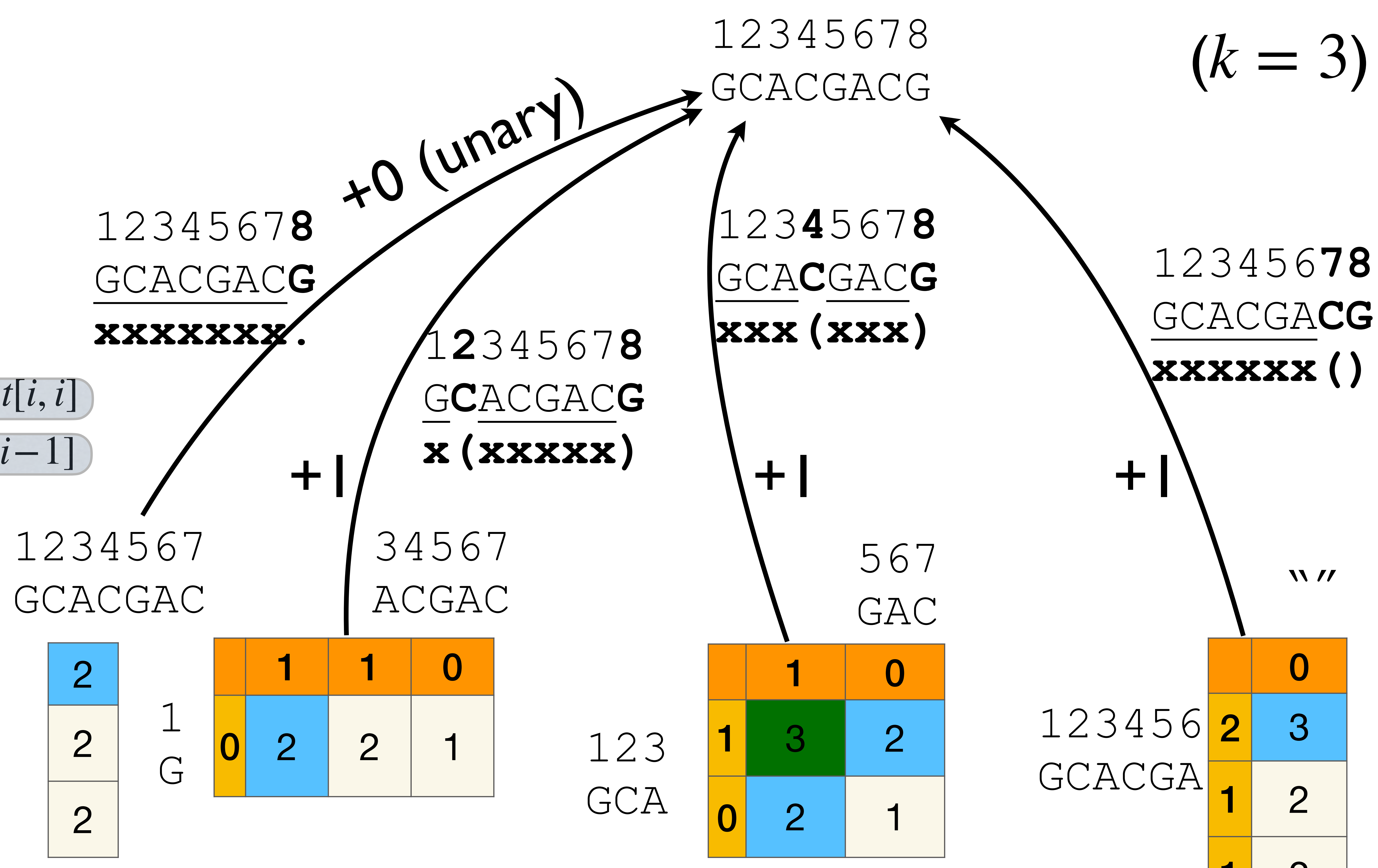
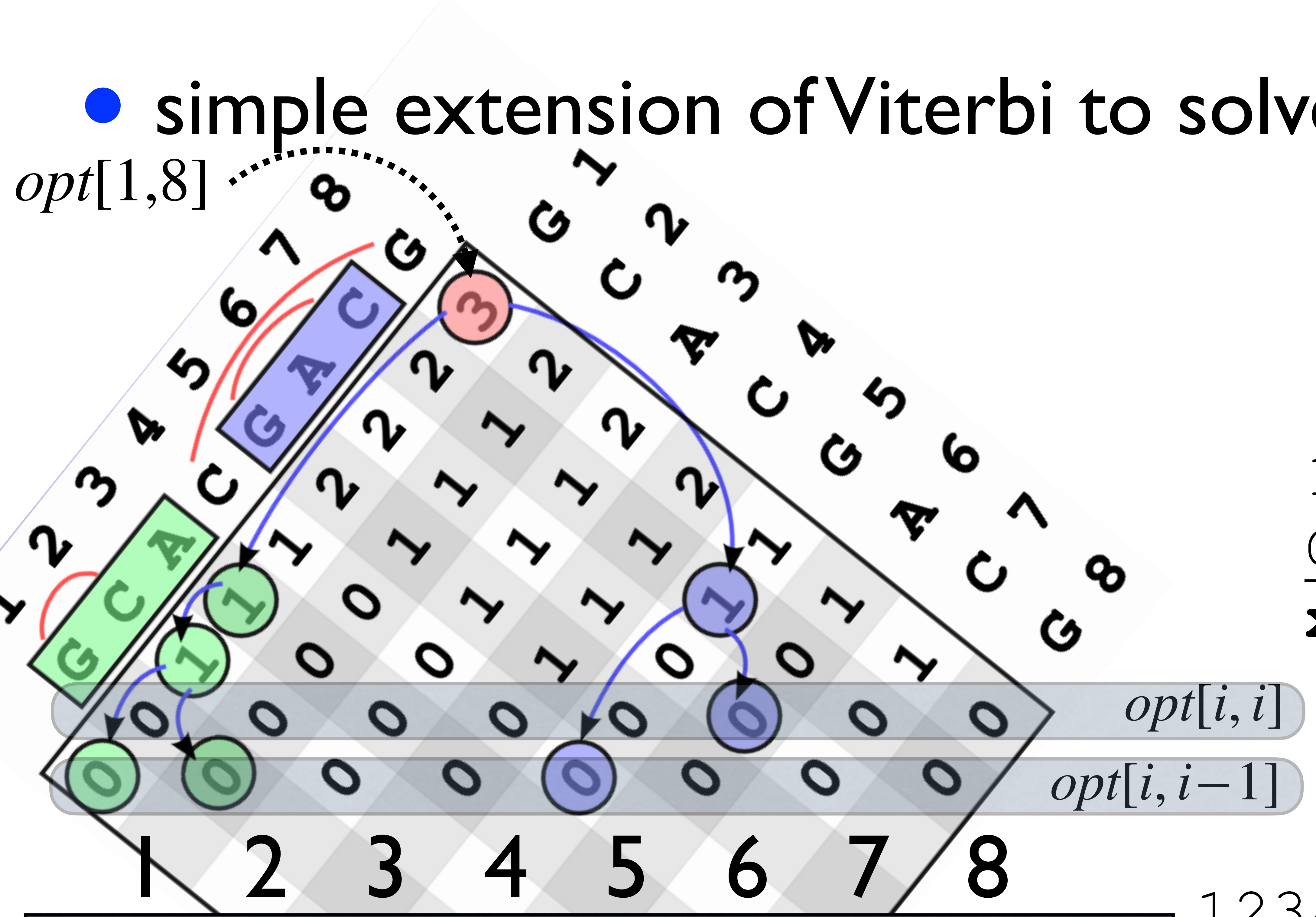
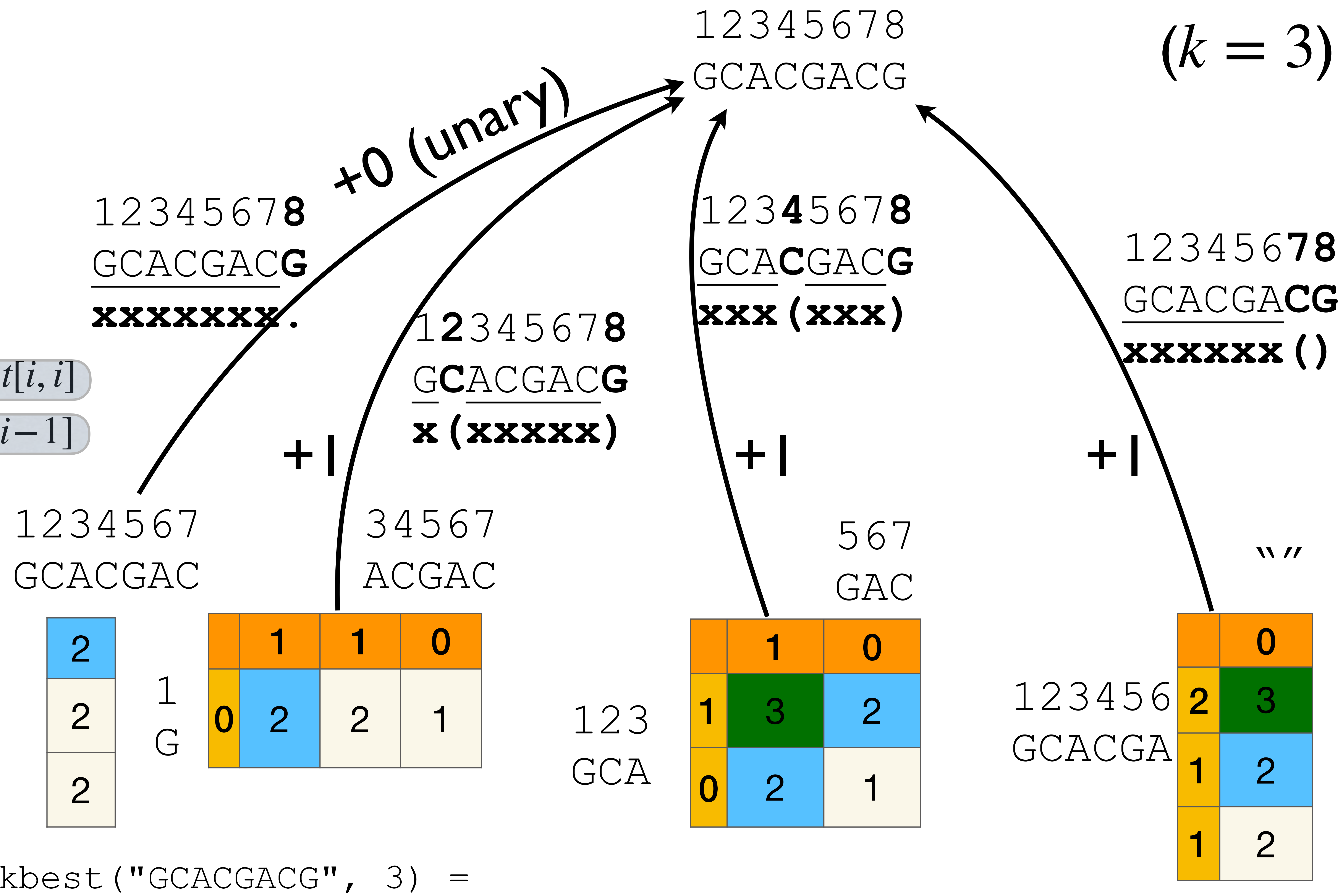



```
kbest("GCACGACG", 3) =
[(3, '().((.))')]
```

k-best Viterbi on Hypergraph

- simple extension of Viterbi to solve k-best on graphs and hyper graphs cf. midterm





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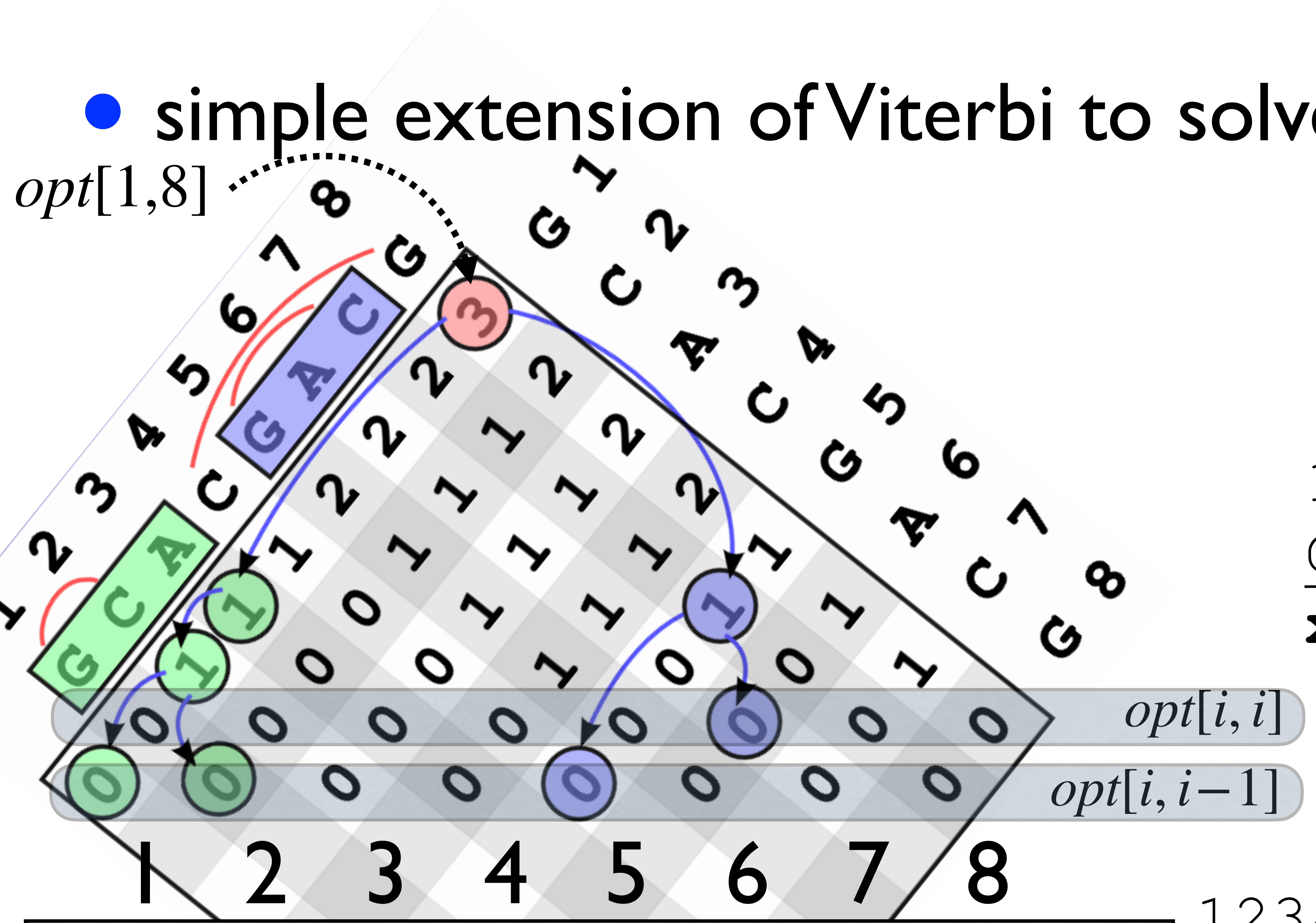
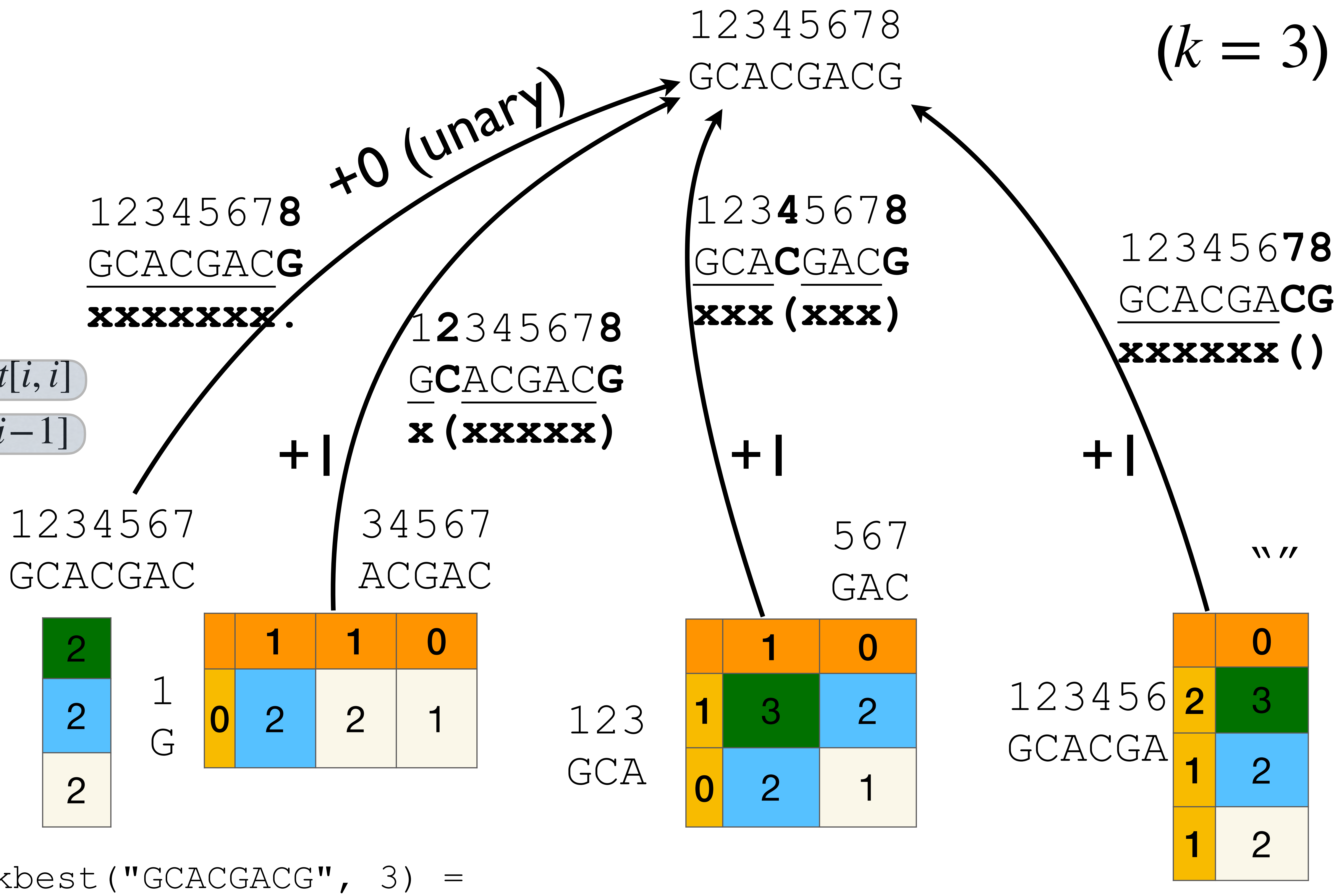
opt	$\oplus$	$\otimes$	$1_{\otimes}$
best	max	+	0
total	+	x	1

```
kbest("GCACGACG", 3) =
[(3, '().((.))'), (3, '().().()')]
```

k-best Viterbi on Hypergraph

- simple extension of Viterbi to solve k-best on graphs and hyper graphs   cf. midterm





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$$opt[i,i] = opt[i,i-1] = 1_{\otimes}$$

opt	$\oplus$	$\otimes$	$1_{\otimes}$
best	max	+	0
total	+	x	1

```
kbest("GCACGACG", 3) =
[(3, '().((.))'), (3, '().().()'), (2, '().()...')]
```