FINITE ELEMENT ANALYSIS SIMPLY EXPLAINED

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1. INTRODUCTION

This article is written for engineers who want to understand how finite element analysis works but are not interested in developing a level of proficiency that would allow them to perform such an analysis themselves. Purists may find many of the explanations to be over-simplified. If this is a concern, you should stop reading now.

Finite element analysis was originally developed for analyzing complex structures. It is currently used to analyze a variety of physical systems including heat transfer, fluid mechanics, magnetism, etc. However, from an intuitive standpoint, the basic ideas are most easily developed using solid mechanics concepts. Most engineering curricula include a course on elementary mechanics of materials. Thus, we will use those concepts as building blocks to illustrate the process. A brief review of some of these basic concepts and matrix mathematics is presented next.

1.1 Elementary Mechanics of Materials

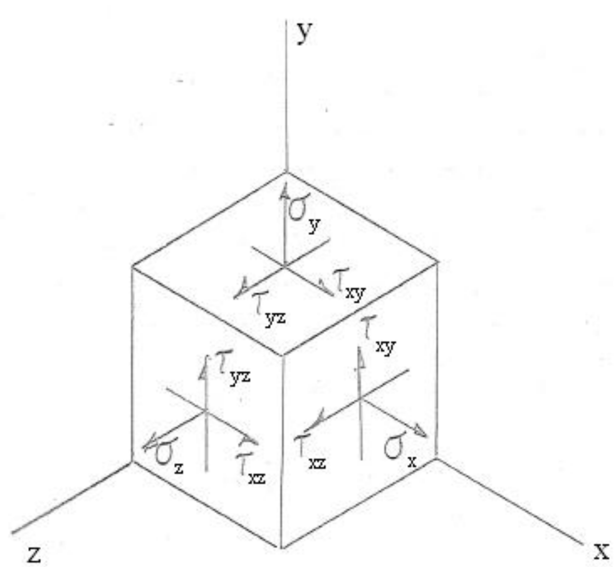
Mechanics of Materials deals with simple structures that deform under load. A body is considered to be in equilibrium when the following is satisfied:

$$\begin{aligned} \sum F_x &= 0 \qquad & \sum M_{x-\text{point}A} &= 0 \\ \sum F_y &= 0 \qquad & \sum M_{y-\text{point}A} &= 0 \\ \sum F_z &= 0 \qquad & \sum M_{z-\text{point}A} &= 0 \end{aligned} \tag{1-1}$$

i.e., when the net forces in the x, y, and z-directions are zero, and the net moments in the x,y, and z-directions about some reference point A are zero.

The effect of applying external loads to a body is to cause stress inside the body. The stresses at an internal point can be represented on the faces of a small cube around the point as shown in Figure 1-1.

Figure 1-1



Note that on each face, there is one component of normal stress acting perpendicular to the face and two components of shear stress acting tangent to the face. The effect of the normal stress is to cause the body to stretch in the direction of the stress and to shrink in the two directions perpendicular to the stress. These deformations can be described by strains (elongation per unit length). The normal strains in the x, y, and z-directions (ε_x, ε_y, ε_z) are related to the normal stresses (σ_x, σ_y, σ_z) through Hooke's law as follows:

$$\begin{aligned} \varepsilon_x &= \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)] \\ \varepsilon_y &= \frac{1}{E}[\sigma_y - \nu(\sigma_x + \sigma_z)] \\ \varepsilon_z &= \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)] \end{aligned} \qquad (1\text{-}2)$$

where E is Young's modulus (or elastic modulus), and υ is Poisson's ratio. E and υ are material properties.

The effect of the shear stress is to cause a shear strain which represents the change in angle (in radians) between the sides of the cube to something smaller or larger than the original right angle. The shear strains (γ_{xy}, γ_{xz}, γ_{yz}) are related to the shear stresses (τ_{xy}, τ_{xz}, τ_{yz},) as follows:

$$\begin{aligned} \gamma_{xy} &= \tau_{xy} / G \\ \gamma_{xz} &= \tau_{xz} / G \\ \gamma_{yz} &= \tau_{yz} / G \end{aligned} \qquad (1\text{-}3)$$

where G is the shear modulus and G=E/(1+2v).

The simple definition of normal strain as stretch per unit length is inconvenient for cases where the strain is not uniform throughout the body. In three dimensions each point on the body will have displacements in the x, y, and z-directions (u, v, and w, respectively). The Theory of Elasticity provides relations between the components of strain and the displacements as

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \varepsilon_y = \frac{\partial v}{\partial y} \qquad \varepsilon_z = \frac{\partial w}{\partial z} \qquad (1\text{-}4)$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \qquad \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \qquad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \qquad (1\text{-}5)$$

1.2 Simultaneous Equations and Matrices

Solving simultaneously linear algebraic equations is a routine task for a computer. Therefore, we are motivated to reduce the mathematics of our physical problem to a set of simultaneous equations.

Let's consider the following set of equations

$$2x + 6y = 10 \qquad (1\text{-}6)$$

$$-3x + 5y = 8 \qquad (1\text{-}7)$$

where x and y are unknowns. We can rewrite these in matrix format as follows:

$$\begin{bmatrix} 2 & 6 \\ -3 & 5 \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} 10 \\ 8 \end{Bmatrix} \qquad (1\text{-}8)$$

On the left side of the equation, the terms in the row of the first matrix are multiplied by the terms in the columns of the second matrix to recover the original equations.
We will find it convenient to use matrix methods to set up the equations for our physical problem for computer solution.

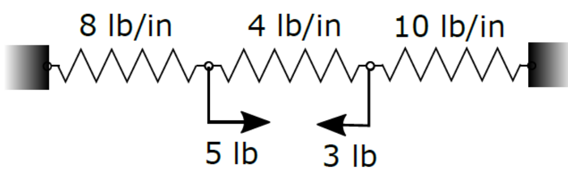
2. MATRIX STRUCTURAL ANALYSIS

Many of the techniques in the finite element procedure are common to those of matrix structural analysis. Therefore, we will review some of these basic concepts.

2.1 Spring Structures

We begin with the analysis of a very simple structure composed of springs (for example , see Figure 2-1).

Figure 2-1



Although these structures are not particularly interesting in a practical sense, their simplicity allows for transparency in the mathematics.

2.1.1 Spring element

Let's consider a simple spring where loads f_i and f_j may be applied to its endpoints (which we will call nodes) and give them the labels i and j as shown in Figure 2-2.

Figure 2-2

When this spring is part of a larger spring structure, we will be interested in the displacements of its node points u_i and u_j. We will use the sign convention that forces and displacements that act to the right are positive and those to the left are negative. Before determining the relationship between the nodal forces f_i and f_j and nodal displacements u_i and u_j, we will use our knowledge that the stretch of the spring must be proportional to the force and vice versa. Therefore, these quantities must be related mathematically as follows

$$K_{ii}u_i + K_{ij}u_j = f_i \qquad (2\text{-}1)$$

$$K_{ji}u_i + K_{jj}u_j = f_j \qquad (2\text{-}2)$$

where K_{ii}, K_{ij}, K_{ji}, and K_{jj} are constants. We will determine these constants by considering two special cases. In the first case we let u_j=0. Then, the equations give

$$K_{ii}u_i = f_i \quad (2\text{-}3)$$

$$K_{ji}u_i = f_j \quad (2\text{-}4)$$

Physically, we have the situation shown in Figure 2-3.

Figure 2-3

The force in the spring is f_i and the compression of the spring is u_i . From the spring law we know

$$f_i = ku_i \quad (2\text{-}5)$$

where k is the spring constant. Comparing this to equation (2-3), we conclude

$$K_{ii} = k \quad (2\text{-}6)$$

The reaction force at node j must be equal and opposite to that at node i. Therefore,

$$f_j = -f_i = -ku_i \quad (2\text{-}7)$$

Comparing this to equation (2-4) we conclude that

$$K_{ji} = -k \quad (2\text{-}8)$$

For the second case, we set u_i=0 . Then the equations (2-1) and (2-2) give

$$K_{ij}u_j = f_i \quad (2\text{-}9)$$

$$K_{jj}u_j = f_j \quad (2\text{-}10)$$

Physically, we have the situation shown in Figure 2-4.

Figure 2-4

The force in the spring is f_j, and the stretch of the spring is u_j . Therefore, the spring law gives

$$f_j = ku_j \quad (2\text{-}11)$$

Comparing this to equation (2-10), we conclude

$$K_{jj} = k \quad (2\text{-}12)$$

The reaction force at node i is equal and opposite to that at node j. Therefore,

$$f_i = -f_j = -ku_j \quad (2\text{-}13)$$

Comparing this to equation (2-9) gives

$$K_{ij} = -k \quad (2\text{-}14)$$

Now that each K has been determined, we can write equations (2-1) and (2-2) as

$$ku_i - ku_j = f_i \quad (2\text{-}15)$$

$$-ku_i + ku_j = f_j \quad (2\text{-}16)$$

Rewriting this in matrix form gives

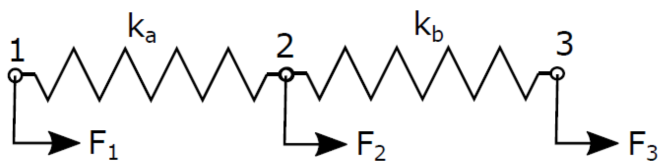
$$\begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix} = \begin{Bmatrix} f_i \\ f_j \end{Bmatrix} \tag{2-17}$$

The first matrix in the equation above is called the element stiffness matrix. As we will see in the next section, we can use this matrix as a building block to determine the equations relating displacements to forces in a system with any number of springs.

2.1.2 System of springs

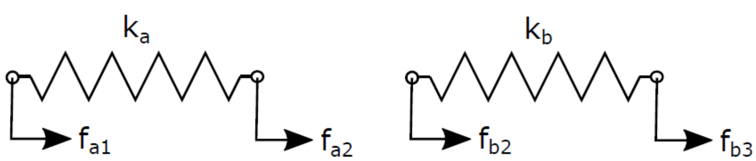
Let us now consider a spring structure composed of two springs a and b with spring constants k_a and k_b as shown in Figure 2-5.

Figure 2-5



It will be helpful to imagine that the external forces F_1, F_2, and F_3 are applied to pins that fit into loops at the ends of the springs, and the pins in turn apply internal forces to the springs. Now let's draw free body diagrams of the individual spring elements as shown in Figure 2-6.

Figure 2-6



The forces shown above are the internal forces applied to the nodes by the pins. Each of these elements essentially replicates the situation from the previous section for a spring element. Therefore, we can write

$$k_a u_1 - k_a u_2 = f_{a1} \tag{2-18}$$

$$-k_a u_1 + k_a u_2 = f_{a2} \tag{2-19}$$

for element a, and

$$k_b u_2 - k_b u_3 = f_{b2} \tag{2-20}$$

$$-k_b u_2 + k_b u_3 = f_{b3} \tag{2-21}$$

for element b.

The next step involves some mathematical slight of hand that may seem contrived. However, the end result will be seen to be beneficial in later steps. We rewrite the equations for element a as follows

$$k_a u_1 - k_a u_2 + 0u_3 = f_{a1} \tag{2-22}$$

$$-k_a u_1 + k_a u_2 + 0u_3 = f_{a2} \tag{2-23}$$

$$0u_1 + 0u_2 + 0u_3 = 0 \tag{2-24}$$

In matrix form of these become

$$\begin{bmatrix} k_a & -k_a & 0 \\ -k_a & k_a & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} f_{a1} \\ f_{a2} \\ 0 \end{Bmatrix} \qquad \text{(2-25)}$$

Next we rewrite the equations for element b as follows

$$0u_1 + 0u_2 + 0u_3 = 0 \qquad \text{(2-26)}$$

$$0u_1 + k_b u_2 - k_b u_3 = f_{b2} \qquad \text{(2-27)}$$

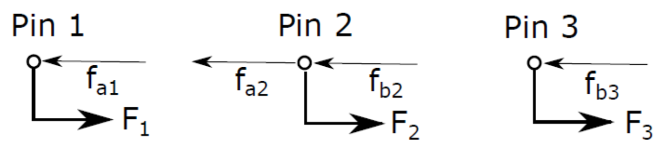
$$0u_1 - k_b u_2 + k_b u_3 = f_{b3} \qquad \text{(2-28)}$$

In matrix form these become

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & k_b & -k_b \\ 0 & -k_b & k_b \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_{13} \end{Bmatrix} = \begin{Bmatrix} 0 \\ f_{b1} \\ f_{b2} \end{Bmatrix} \qquad \text{(2-29)}$$

Now let's draw free body diagrams of the individual pins which are subjected to the external forces and the reactions from the internal forces on the elements as show in Figure 2-7.

Figure 2-7



Force equilibrium gives the following relations

$$F_1 = f_{a1} \qquad \text{(2-30)}$$

$$F_2 = f_{a2} + f_{b2} \qquad \text{(2-31)}$$

$$F_3 = f_{b3} \qquad \text{(2-32)}$$

We can rewrite these in matrix form as follows

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} f_{a1} \\ f_{a2} + f_{b2} \\ f_{b3} \end{Bmatrix} = \begin{Bmatrix} f_{a1} \\ f_{a2} \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ f_{b2} \\ f_{b3} \end{Bmatrix} \qquad \text{(2-33)}$$

Note that the last two matrices are identical to those in equations (2-25) and (2-29). Therefore we can write

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{bmatrix} k_a & -k_a & 0 \\ -k_a & k_a & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & k_b & -k_b \\ 0 & -k_b & k_b \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \qquad \text{(2-34)}$$

Rearranging terms gives

$$\begin{bmatrix} k_a & -k_a & 0 \\ -k_a & k_a + k_b & -k_b \\ 0 & -k_b & k_b \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} \qquad \text{(2-35)}$$

The first matrix is called the structure stiffness matrix. It relates the nodal displacements to the external loads for the two-spring structure. We observe that it can be obtained by combining the element stiffness matrices in a specific pattern that is easy to generalize. For the general case shown in Figure 2-8,

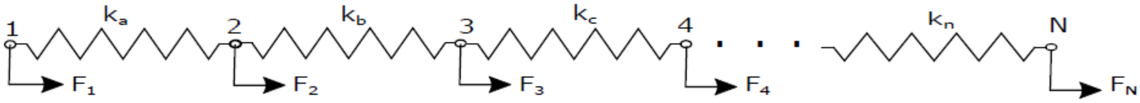


Figure 2-8

we can write

$$\begin{bmatrix} k_a & -k_a & 0 & 0 & . & . & . \\ -k_a & k_a+k_b & -k_b & 0 & . & . & . \\ 0 & -k_b & k_b+k_c & -k_c & . & . & . \\ 0 & 0 & -k_c & k_c+k_d & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & -k_n \\ . & . & . & . & . & -k_n & -k_n \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ . \\ . \\ u_N \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ . \\ . \\ F_N \end{Bmatrix} \qquad (2\text{-}36)$$

Example: Find the nodal displacements and reaction forces for the spring system in Figure 2-9.

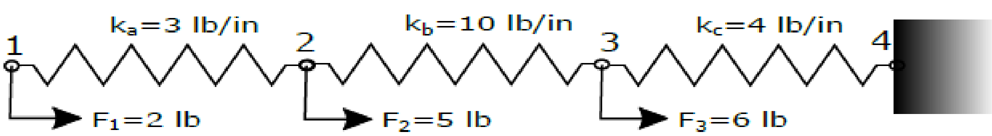


Figure 2-9

Using the pattern developed above, the system of equations becomes

$$\begin{bmatrix} 3 & -3 & 0 & 0 \\ -3 & 3+10 & -10 & 0 \\ 0 & -10 & 10+4 & -4 \\ 0 & 0 & -4 & 4 \end{bmatrix} \begin{Bmatrix} u_1 = ? \\ u_2 = ? \\ u_3 = ? \\ u_4 = 0 \end{Bmatrix} = \begin{Bmatrix} F_1 = 2 \\ F_1 = 5 \\ F_1 = 6 \\ F_4 = ? \end{Bmatrix} \qquad (2\text{-}37)$$

We have four equations for the four unknowns which can be solved by computer.

3. FINITE ELEMENT ANALYSIS

A finite element analysis involves treating a structure as a collection of elements connected at node points. Within each element the response is assumed to follow a simple mathematical form which allows the formation of the element stiffness matrix in a straightforward manner. The matrix structural analysis techniques of the previous section are then used to calculate the response of the whole structure.

3.1 Energy Approach

The most common approach to enforce equilibrium in a body is the requirement of satisfying Newton's laws. However, an alternative approach involving energy is more convenient for obtaining the element stiffness matrix. We begin by defining strain energy. Recall the situation of a simple spring as shown in Figure 2-10.

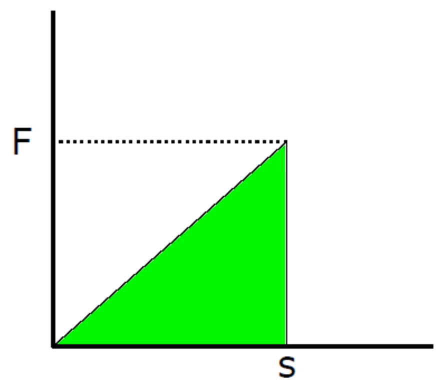
Figure 2-10

The relationship between the force F and the stretch s is

$$F = ks \tag{3-1}$$

A plot of force versus stretch produces a straight line as shown in Figure 2-11.

Figure 2-11



The work done by the force is equal to the area under the curve; i.e.,

$$W = \frac{1}{2}Fs = \frac{1}{2}(ks)s = \frac{1}{2}ks^2 \tag{3-2}$$

The work done by the force is stored in the spring as strain energy U where

$$U = \frac{1}{2}ks^2 \tag{3-3}$$

We also define the potential V of the external force as the negative of the product of the force times the displacement in the direction of the force.

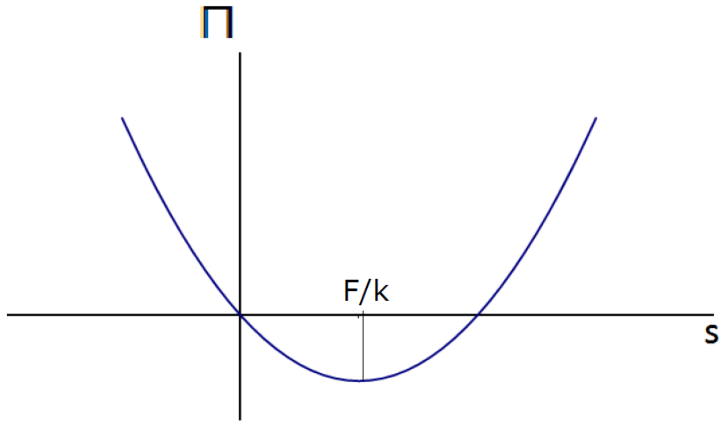
$$V = -Fs \tag{3-4}$$

The total potential is defined as the sum of the strain energy and potential of the external force.

$$\Pi = U + V = \frac{1}{2}ks^2 - Fs \tag{3-5}$$

A plot of Π versus s is shown Figure 2-12.

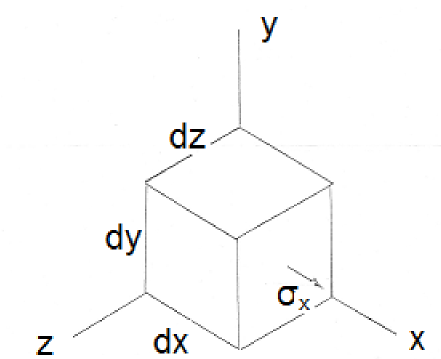
Figure 2-12



This curve has a minimum at s=F/k which is the value that s takes on when the spring is in equilibrium. This is not a coincidence. There is a principle of mechanics that states that the total potential of a structure is a minimum when it is in equilibrium. Henceforth, we will use this principle to develop the stiffness matrix for an element.

We need an expression for strain energy in a solid body. Consider a small block with dimensions *dx* by *dy* by *dz* inside a solid with a normal stress σ_x in the x-direction as shown in Figure 2-13.

Figure 2-13



The net force F_x on the block in the x-direction is the stress times the area dydz.

$$F_x = \sigma_x dydz \tag{3-6}$$

The stretch of the block in the x-direction is the strain ε_x times the length dx.

$$s = \varepsilon_x dx \tag{3-7}$$

As in the case of a spring, the strain energy in the block dU is

$$dU = \frac{1}{2} F_x s = \frac{1}{2} (\sigma_x dydz)(\varepsilon_x dx) = \frac{1}{2} \sigma_x \varepsilon_x dxdydz \qquad (3\text{-}7)$$

We note that dxdydz equals the volume of the block dvol so that

$$dU = \frac{1}{2} \sigma_x \varepsilon_x dvol \tag{3-8}$$

The strain energy for the whole body will consist of a summation (integration) of the strain energy in all of the blocks that make up the body; i.e.,

$$U = \iiint \frac{1}{2}\sigma_x \varepsilon_x dvol \tag{3-9}$$

For general three-dimensional loading, all the stresses and strains will contribute to the strain energy in the body. Therefore, the general expression for strain energy is

$$U = \iiint \frac{1}{2}(\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \sigma_z \varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{xz}\gamma_{xz} + \tau_{yz}\gamma_{yz})dvol \tag{3-10}$$

Also, for the general case, there will be multiple external loads $\vec{F}_i$ experiencing displacements $\vec{u}_i$ which leads to the potential of the external loads as

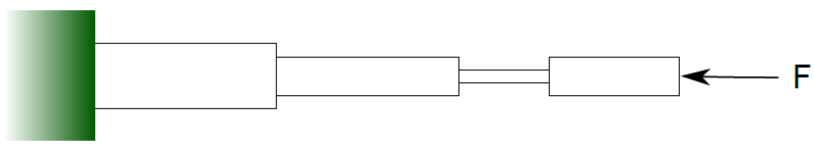
$$V = -\sum_{i=1}^{N} \vec{F}_i \cdot \vec{u}_i \tag{3-11}$$

The total potential Π will again be the sum of U and V.

3.2 Bar Element

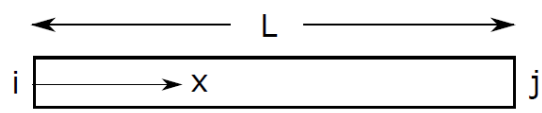
We will now apply the principles of the previous section to develop the stiffness matrix for a bar element. This element will be used to analyze structures consisting of bars under axial loading like the one shown Figure 2-14.

Figure 2-14



Let's examine one of these bar elements with an x-axis indicating a position on the bar relative to the left end as shown Figure 2-15.

Figure 2-15



As part of a larger structure, we expect that points on this bar will have a horizontal displacement u, and the amount of displacement in the bar will vary from point to point between the left end (node i) and in the right end (node j). We want to represent this variation mathematically. Therefore, we will pick a mathematical function that we think will be a reasonable representation of this, even if it may not be the exact one. In making this selection, we seek a balance between the desire for mathematical simplicity and the need for replicating what is happening physically with reasonable accuracy. Let's assume that the horizontal displacement in the element can be represented with a first order polynomial (a straight line); i.e.,

$$u = a_0 + a_1 x \tag{3-12}$$

The values for a_0 and a_1 can be determined by the following requirements:

At $x = 0,\ u = u_i$ (3-13)

At $x = L,\ u = u_j$ (3-14)

Thus, we obtain

$$u = u_i + \frac{(u_j - u_i)}{L} x \tag{3-15}$$

This step will always be the starting point in the process of determining the element stiffness matrix; i.e., we will choose mathematical functions that we expect have the capability of representing the displacements in the element, at least approximately. Next, we will determine the strain energy in the bar element. We start by finding the axial strain. Rather than using the simple definition of the normal strain as the stretch of the bar divided by the original length of the bar, we will use the more general definition from the Theory of Elasticity (Section 1) which states that the normal strain is the rate of change of displacement with respect to position; i.e.,

$$\varepsilon_x = \frac{\partial u}{\partial x} \tag{3-16}$$

Using this on equation (3-15) gives

$$\varepsilon_x = \frac{(u_j - u_i)}{L} \tag{3-17}$$

which agrees with the simplified definition.

Since the only nonzero stress in the bar is σ_x, the strain energy is

$$U = \iiint \frac{1}{2} \sigma_x \varepsilon_x dvol \tag{3-18}$$

The stress σ_x can be related to the strain by Hooke's law giving

$$\sigma_x = E\varepsilon_x = E\frac{(u_j - u_i)}{L} \tag{3-19}$$

The strain energy becomes

$$U = \iiint \frac{1}{2} \left[E\frac{(u_j - u_i)}{L} \right] \left[\frac{(u_j - u_i)}{L} \right] dvol \tag{3-20}$$

$$U = \frac{1}{2} \frac{E}{L^2} (u_j^2 - 2u_j u_i + u_i^2) \iiint dvol \tag{3-21}$$

The integral is easy to evaluate and is simply equal to the volume of the bar; i.e.,

$$vol = AL \tag{3-22}$$

where A is the cross-section area. Therefore,

$$U = \frac{1}{2} \frac{EA}{L} (u_j^2 - 2u_j u_i + u_i^2) \tag{3-23}$$

Next, we will find potential of the applied loads. If we have horizontal forces f_i and f_j acting at nodes i and j respectively, using equation (3-11) gives

$$V = -f_i u_i - f_j u_j \tag{3-24}$$

This gives the total potential as

$$\Pi = U + V = \frac{1}{2} \frac{EA}{L} (u_j^2 - 2u_j u_i + u_i^2) - f_i u_i - f_j u_j \tag{3-25}$$

Now we apply the principle of minimum total potential which for this case states that the correct values of u_i and u_j are those that make Π a minimum. Recall from second year calculus that a minimum of a function of several variables can be found by differentiating the function with respect to each variable and setting the derivative equal to zero. For our case this gives

$$\frac{\partial \Pi}{\partial u_i} = \frac{1}{2}\frac{EA}{L}(-2u_j + 2u_i) - f_i = 0 \tag{3-26}$$

$$\frac{\partial \Pi}{\partial u_j} = \frac{1}{2}\frac{EA}{L}(2u_j - 2u_i) - f_j = 0 \tag{3-27}$$

Rearranging gives

$$\frac{EA}{L}u_i - \frac{EA}{L}u_j = f_i \tag{3-28}$$

$$-\frac{EA}{L}u_i + \frac{EA}{L}u_j = f_j \tag{3-29}$$

Rewriting this in matrix form gives the element stiffness matrix from the relation

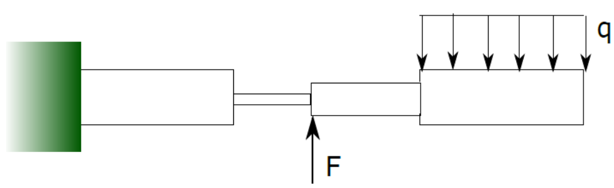
$$\begin{bmatrix} \frac{EA}{L} & -\frac{EA}{L} \\ -\frac{EA}{L} & \frac{EA}{L} \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix} = \begin{Bmatrix} f_i \\ f_j \end{Bmatrix} \tag{3-30}$$

Comparing this to equation (2-17) for a spring, we see that the bar element is identical to a spring element with stiffness k=EA/L. This new approach for finding the stiffness matrix may seem unnecessarily complicated compared to the relatively simple steps used for the spring element. However, as elements increase in complexity, this new approach that begins with the selection of a displacement function is the only tractable one. With the element stiffness matrix for a bar element, we can use matrix structural analysis methods of the previous section to analyze bar structures.

3.3 Beam Element

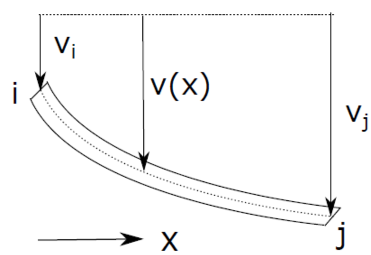
Now we will examine beam structures subjected to transverse forces as shown in Figure 2-16.

Figure 2-16



Consider a beam element which experiences transverse displacement v(x) as shown in Figure 2-17.

Figure 2-17



As we did for the bar element, we need to select a mathematical function to represent the displacement v as a function of position x. For the bar element, we chose a first order polynomial (a straight-line function). However, we expect the beam to deflect in a curved shape which makes the straight line poor choice. It turns out that the ideal choice for a beam element is a third order (cubic) polynomial.

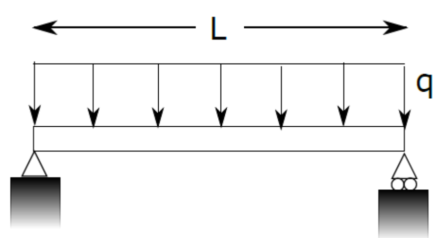
$$v = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \qquad (3\text{-}31)$$

Could we have chosen a different type of polynomial or a different class of functions (e.g., trigonometric)? In principle, the answer is yes. However, the mathematics works out perfectly for the third order polynomial. In fact, for most elements, there is an ideal choice for the displacement function that makes all of the mathematics work out in a very convenient manner.

The rest of the steps for determining the element stiffness matrix follow the same pattern as that for the bar element. Once the element stiffness matrix is known, constructing the structure stiffness matrix and solving for the response of the beam structure follows the same logic as described in Section 3.2. If there are distributed loads acting on the beam, they are replaced by approximately equivalent concentrated loads acting at node points.

Simple beam structure problems can be solved by hand using Elementary Mechanics of Materials methods. Does the finite element solution match the exact hand solution? The answer depends on how well the cubic polynomial used for the displacement function matches the exact solution. Beams subjected to concentrated loads deflect in the shape of a cubic polynomial. In this case, the finite element results will match exactly. Beams subjected to distributed loads generally do not deflect in the shape of a cubic polynomial, and the finite element solution will be approximate. To see this, consider the very simple case of simply supported beam under a uniformly distributed load as shown in Figure 2-18.

Figure 2-18



Elementary beam theory gives the beam displacement in the form of a fourth order polynomial and the center deflection as

$$v = 0.0130\,\frac{qL^4}{EI} \qquad (3\text{-}32)$$

If we model the game with a single finite element, we get the center deflection as

$$v_{FE} = 0.0104\,\frac{qL^4}{EI} \qquad (3\text{-}33)$$

which shows the limitations of the cubic polynomial in matching the correct center deflection. If we divide the beam into two equal length elements, we get the center deflection as

$$v_{FE} = 0.0130 \frac{qL^4}{EI} \tag{3-34}$$

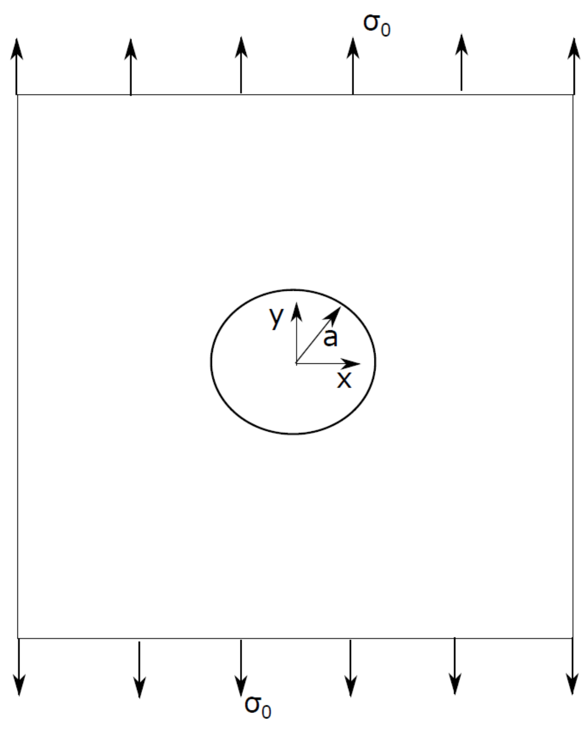
By refining the model with more elements we are able to get a more exact solution. The effects of model refinement will be discussed in more detail in the next section.

So far we have looked at structures that are effectively one-dimensional; i.e., all of the beams/bars are in a single horizontal line. If we want to model a bicycle frame, for example, we will need three-dimensional versions of these elements. The process for extending these elements to two or three dimensions is straightforward but tedious and will not be done here. The logic for assembling the structure stiffness matrix from the element stiffness matrices is also more complex.

3.4 Two-Dimensional Solid Body Element

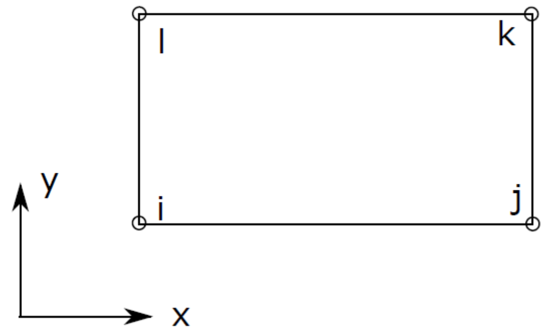
So far**,** we have examined the analysis of structures made of a collection of “sticks.” The elements used in these stick-like structures would clearly not work for the case shown in Figure 2-19

Figure 2-19



which consists of a rectangular plate under tension stress σ_0 with a circular hole with radius a at the center. Suppose we were to divide this plate into a number of rectangular plate elements, like that shown in Figure 2-20 with node points at the corners and edges parallel to the x and y axes. In general, points on the plate experience a horizontal displacement u and a vertical displacement v. These displacements will vary with position on the plate. Therefore, they will be functions of both x and y.

Figure 2-20



As before, our starting point for determining the element stiffness matrix is to select mathematical functions that can, at least approximately, represent the displacements in the element. It turns out that the mathematics works out very conveniently if the following displacement functions are chosen.

$$u(x, y) = a_0 + a_1 x + a_2 y + a_3 xy \quad (3\text{-}35)$$

$$v(x, y) = b_0 + b_1 x + b_2 y + b_3 xy \quad (3\text{-}36)$$

From elasticity theory, the strains can be obtained from the derivatives of the displacements as follows.

$$\varepsilon_x = \frac{\partial u}{\partial x} = a_1 + a_3 y \quad (3\text{-}37)$$

$$\varepsilon_y = \frac{\partial v}{\partial y} = b_2 + b_3 x \quad (3\text{-}38)$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = a_2 + a_3 x + b_1 + b_3 y \quad (3\text{-}39)$$

Hooke's law for the two-dimensional stress case gives the stresses in terms of the strains as

$$\sigma_x = \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y) \quad (3\text{-}40)$$

$$\sigma_y = \frac{E}{1-\nu^2}(\varepsilon_y + \nu\varepsilon_x) \quad (3\text{-}41)$$

$$\tau_{xy} = \frac{E}{(1+2\nu)}\gamma_{xy} \quad (3\text{-}42)$$

Substituting equations (3-37) to (3-39) into equations (3-40) to (3-42) gives the stresses as

$$\sigma_x = \frac{E}{1-\nu^2}[a_1 + a_3 y + \nu(b_2 + b_3 x)] \quad (3\text{-}43)$$

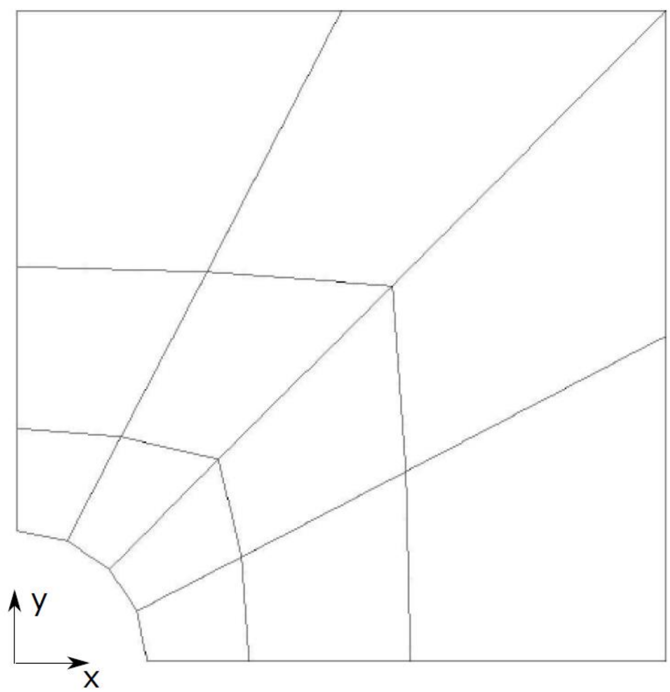
$$\sigma_y = \frac{E}{1-\nu^2}[b_2 + b_3 x + \nu(a_1 + a_3 y)] \quad (3\text{-}44)$$

$$\tau_{xy} = \frac{E}{(1+2\nu)}(a_2 + a_3 x + b_1 + b_3 y) \quad (3\text{-}45)$$

We observe that our choice of element displacement functions leads to stresses that are assumed to vary linearly (i.e., in a straight-line) from point to point inside the element. This assumption will affect the manner in which we divide up the plate into elements.

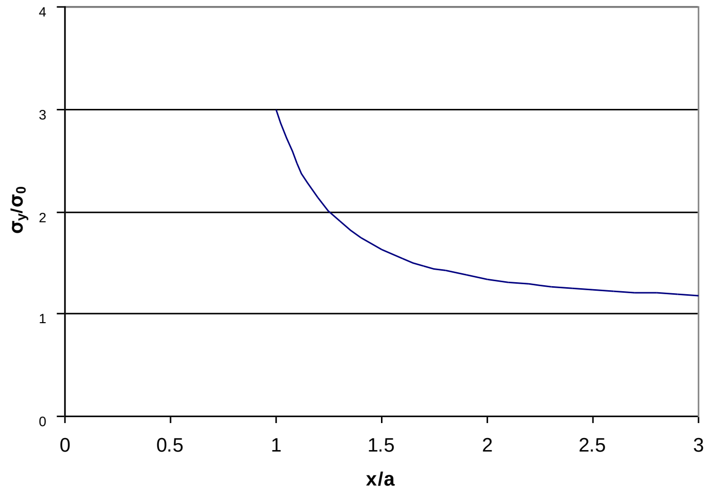
The process of determining the element stiffness matrix proceeds as before with the application of the minimum total potential principle. Before proceeding to this step, it is probably clear to the reader that a rectangular element will have a serious problem representing the geometry around the circular hole. In fact, any geometry where the edges of the body do not run parallel to x or y will be problematic. Therefore, before proceeding, the next step is typically to generalize the rectangular shape into a quadrilateral shape through a coordinate transformation. The mathematics for accomplishing this is straightforward but somewhat tedious. The end result is that we have an element layout (mesh) that might look like that in Figure 2-21 for the upper right quadrant of the plate for the case where the width of the plate is 5 times the diameter.

Figure 2-21



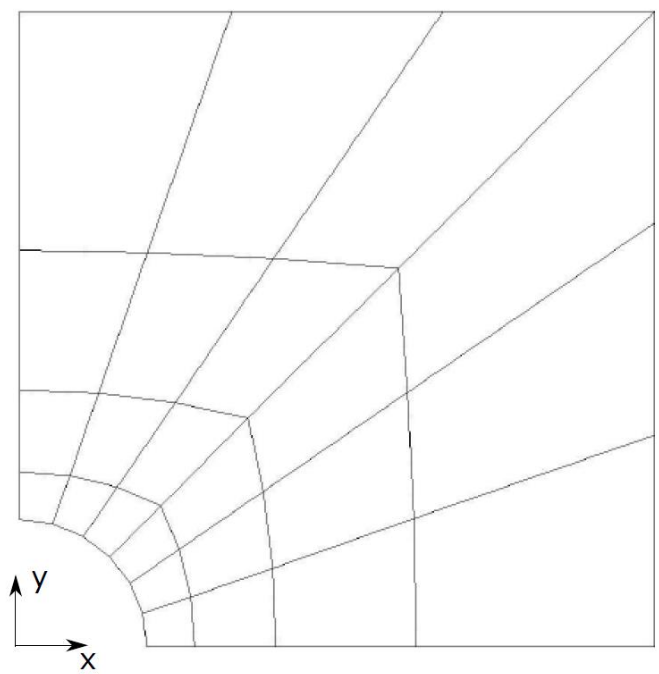
Will this mesh provide accurate analysis results? To answer this question, it is helpful to have some idea of how the stress distribution should look. There is an analytical solution to this problem for the case of the plate being very large (effectively infinite) compared to the size of the hole. The stress σ_y/σ_0 is plotted as a function of position x/a near the edge of the hole in Figure 2-22.

Figure 2-22



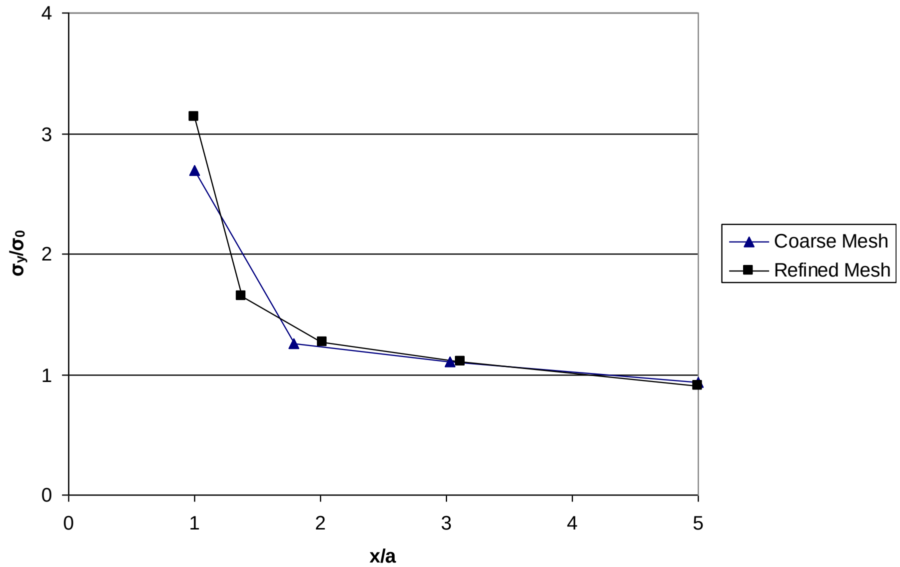
We see a rapid rise in stress as we approach the hole. In this element the stress is assumed to vary linearly across the element. The mesh shown in Figure 2-21 appears that it may be too coarse to represent a rapidly varying stress. It is interesting to compare the results from this mesh to the results from the more refined one shown in Figure 2-23.

Figure 2-23



After performing an analysis using the commercial program ANSYS, we can make a plot of stress σ_y/σ_0 versus position x/a for each mesh as shown in Figure 2-24.

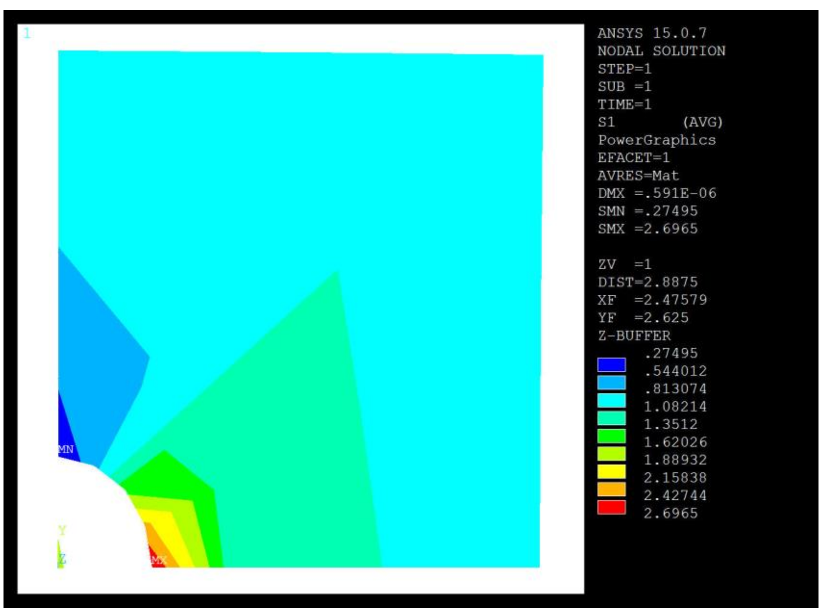
Figure 2-24



We see that the refined mesh gives a steeper stress gradient than the coarse mesh. If we look up this case in a handbook for stress concentration factors, it would tell us that the stress at the edge of the hole should be $\sigma_y/\sigma_0=3.14$. The refined mesh gives a value very close to this, but the coarse mesh is low by about 14%. In general, as the mesh is refined, the finite element solution will converge to the correct solution.

What would we have done if we had no idea how the stress distribution should look? In this case, we could have used the coarse mesh results to construct a contour map of the maximum principal stress as shown in Figure 2-25.

Figure 2-25



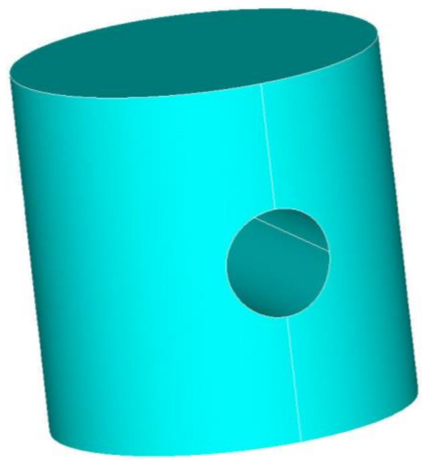
This contour map is similar to a contour map for ground elevation except that, in this case, various stress levels are represented by different colors. We see quite clearly that there is a steep gradient in stress near the hole, which will require a fine mesh in this area

There is another version of this quadrilateral element that uses higher order polynomials for the displacement functions. To make the mathematics work out, additional nodes need to be added to the mid-side of the edges of the element. This element allows for a coarse mesh to give accurate results, but the total number of equations to be solved does not decrease because there are eight nodes per element in this element, as opposed to four nodes per element is in the lower order element.

3.5 Three-Dimensional Solid Body Element

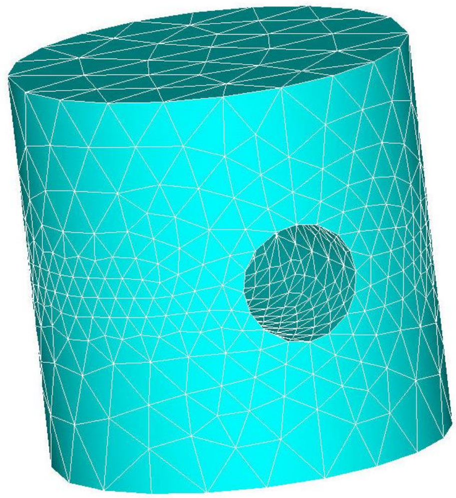
Suppose we wanted to determine the stresses in a solid cylinder with a hole running perpendicular to the central axis as shown in Figure 2-26.

Figure 2-26



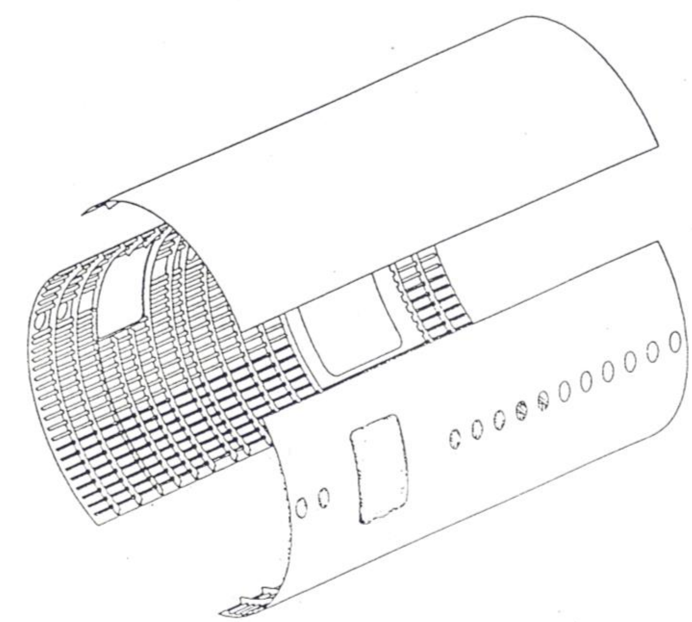
We would need to add a third dimension to the quadrilateral element in the previous section to create a brick-like hexahedral element (i.e., a block whose sides are not necessarily at right angles to each other) for three-dimensional bodies. Complex three-dimensional shapes can be challenging to mesh using this element. In these cases, a tetrahedral element (i.e., a pyramid-like element with four triangular-shaped sides) can be used. The mesh of the cylinder using this element is shown in Figure 2-27.

Figure 2-27



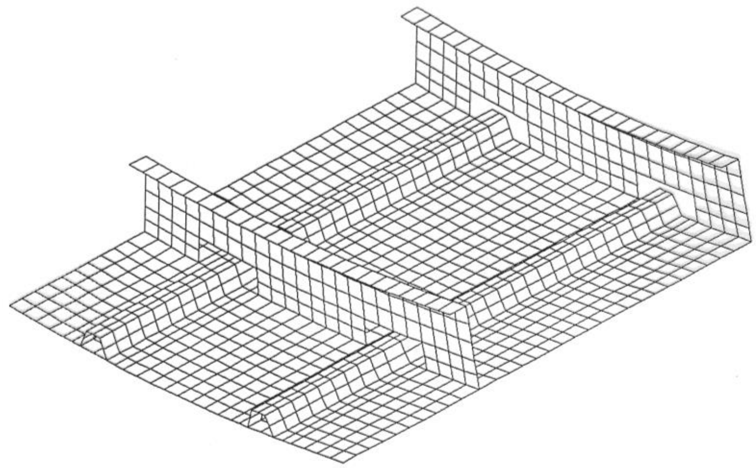
For bodies that consists of thin-walled three-dimensional surfaces like the section of an aircraft fuselage shown in Figure 2-28,

Figure 2-28



a quadrilateral shell element can be used. This element has the same properties as the two-dimensional quadrilateral element discussed in the previous section, but with the added capability for allowing out-of-plane displacement and bending. A mesh of a section of fuselage using this element is shown in Figure 2-29.

Figure 2-29



3.6 Nonlinear and Dynamic Effects

Nonlinear effects can be a complicating factor. If we re-examine the bar element, we observe that the element stiffness matrix contains area A, modulus E, and length L. If the element experiences a large deformation, the change in length could be large enough that using the original length in the calculations would lead to significant inaccuracy. In this case, the length L becomes a function of nodal displacements. This will change the resulting equations from linear algebraic equations to nonlinear algebraic equations. Nonlinear material behavior (e.g., plasticity) will also cause the resulting equations to become nonlinear. These equations can be tricky to solve and often require some finesse on the part of the analyst to arrive at a converged solution.

Dynamic effects from structural vibrations or impulsive loading can also complicate the analysis. In this case Newton's law contains the inertia term $m\vec{a}$ in $\vec{F} = m\vec{a}$. This results in a need for an element mass matrix as well as a stiffness matrix. The system equations become differential equations which are considerably more challenging to solve then linear algebraic equations.

3.7 Non-Structural Applications

Finite element analysis can be applied to a variety processes outside of solid mechanics. In these cases, the development of a minimization principle tends to be more mathematically abstract than was the case for solid mechanics and will not be described here.

As an example, suppose we replace the stress analysis problem depicted in Figure (2-19) with a steady-state heat conduction problem where surface temperature T_s or surface heat flux q_s is prescribed around the boundary of the plate. The correct internal temperature distribution is the one that causes the potential

$$\Pi = \iiint \frac{1}{2}\left[\left(\frac{\partial T}{\partial x^2}\right)^2 + \left(\frac{\partial T}{\partial y^2}\right)^2\right] dvol - \oint T_s q_s ds \tag{3-47}$$

to be a minimum. We would proceed in exactly the same manner as that for the quadrilateral element by assuming a mathematical function to represent the temperature inside the element as

$$T(x, y) = a_0 + a_1 x + a_2 y + a_3 xy \tag{3-48}$$

Using this we would develop an equivalent "stiffness" matrix following the same procedures as described earlier.

4. CONCLUSION

Finite element analysis provides a means of conducting a computer simulation of the response of a body to external stimulation. The overall response of the body may be difficult or impossible to be described mathematically in closed form. In a finite element analysis, the body is imagined to be composed of a number of smaller elements where the response inside a given element can be approximated by simple mathematical expressions. Commercial finite element programs are available for analyzing a variety of physical phenomena. The creators of these programs have made great strides in making these programs sufficiently "idiot-proof" that almost anyone with a minimal amount of training can perform an analysis and get "results." However, it should be kept in mind that a finite element analysis merely provides an approximate solution to a mathematical problem. Sound engineering judgment will always be required to reduce a physical problem to an idealized case they can be successfully modeled and to determine if the results of the analysis provide a realistic representation of the original physical system.