Baseband Transmission of Binary Signals

Let $g_i(t)$, $i = 1, 2$, be a signal transmitted over an AWGN channel. Consider the following receiver.

$$x(t) = g_i(t) + W(t), \ (n-1)T \leq t \leq nT, \ n = 1, 2, \ldots$$

$W(t)$ is a zero-mean white Gaussian noise process with psd $\frac{N_0}{2}, \forall f$
At the output of the LTI filter,

\[ y(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau = \int_{-\infty}^{\infty} [g_i(\tau) + W(\tau)] h(t - \tau) d\tau \]

\[ y(t) = \int_{-\infty}^{\infty} g_i(\tau) h(t - \tau) d\tau + \int_{-\infty}^{\infty} W(\tau) h(t - \tau) d\tau. \]

At the sampling instant \( t = nT \),

\[ y(nT) = \int_{-\infty}^{\infty} g_i(\tau) h(nT - \tau) d\tau + \int_{-\infty}^{\infty} W(\tau) h(nT - \tau) d\tau. \]

Let \( g_1(t) \) be the transmitted pulse when a logical “1” is sent and let \( g_2(t) \) be the transmitted pulse when a logical “0” is sent.

Then a possible decision strategy is:

If \( y(nT) > A \), then \( g_1(t) \) was transmitted (“1” sent)
If \( y(nT) \leq A \), then \( g_2(t) \) was transmitted (“0” sent)
Since $h(t)$ is LTI, the observation $y(nT)$, given the knowledge of $g_i(t)$, is a Gaussian random variable. Hence, to establish the decision criterion we only need to compute the conditional mean and conditional variance of $y(nT)$ given that we know $g_i(t)$.

The conditional expected value (mean) of the filter output given that $g_i(t)$ was sent is given by

$$
E\{y(nT)|g_i(t)\} = E\left\{\int_{-\infty}^{\infty} g_i(\tau)h(nT-\tau)d\tau | g_i(t)\right\} + E\left\{\int_{-\infty}^{\infty} W(\tau)h(nT-\tau)d\tau | g_i(t)\right\}
$$

$$
= \int_{-\infty}^{\infty} E\{g_i(\tau)h(nT-\tau)|g_i(t)\}d\tau + \int_{-\infty}^{\infty} E\{W(\tau)h(nT-\tau)|g_i(t)\}d\tau
$$

$$
= \int_{-\infty}^{\infty} g_i(\tau)h(nT-\tau)d\tau + \int_{-\infty}^{\infty} \underbrace{E\{W(\tau)\}}_{0} h(nT-\tau)d\tau
$$

$$
= \int_{-\infty}^{\infty} g_i(\tau)h(nT-\tau)d\tau \equiv G_i, \ i = 1, 2
$$
Moreover, the conditional variance is

\[ \text{Var}(y(nT) | g_i(t)) = E \left\{ \left| y(nT) - E \left\{ y(nT) \big| g_i(t) \right\} \right|^2 | g_i(t) \right\} \]

\[ = E \left\{ \left| \int_{-\infty}^{\infty} W(\tau) h(nT - \tau) d\tau \right|^2 | g_i(t) \right\} = E \left\{ \int_{-\infty}^{\infty} W(\tau) h(nT - \tau) d\tau \right|^2 \}

\[ = E \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W(\tau) W^*(\lambda) h(nT - \tau) h^*(nT - \lambda) d\tau d\lambda \right\}

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E\left\{ W(\tau) W^*(\lambda) \right\} h(nT - \tau) h^*(nT - \lambda) d\tau d\lambda \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{N_0}{2} \delta(\tau - \lambda) h(nT - \tau) h^*(nT - \lambda) d\tau d\lambda \]

\[ = \frac{N_0}{2} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \delta(\tau - \lambda) h(nT - \tau) d\tau \right) h^*(nT - \lambda) d\lambda \]

\[ = \frac{N_0}{2} \int_{-\infty}^{\infty} |h(nT - \lambda)|^2 d\lambda = \frac{N_0}{2} \int_{-\infty}^{\infty} |h(\lambda)|^2 d\lambda \]

\[ = \frac{N_0}{2} \int_{-\infty}^{\infty} |H(f)|^2 df = \sigma_N^2 \]
Therefore,

\[
f(y(nT)|"1" \text{ sent}) = \frac{1}{\sqrt{2\pi}\sigma_N} e^{-(y-G_1)^2/2\sigma_N^2}
\]

\[
f(y(nT)|"0" \text{ sent}) = \frac{1}{\sqrt{2\pi}\sigma_N} e^{-(y-G_2)^2/2\sigma_N^2}.
\]

Therefore, an error will occur if either a) choose “0” when “1” was sent, or b) choose “1” when “0” was sent. Mathematically,

\[
P\{\text{choose "0" |"1" was sent}\} = P\{y(nT) \leq A |"1" \text{ was sent}\}
\]

\[
= \int_{-\infty}^{A} f(y(nT)|"1" \text{ was sent})dy
\]

and

\[
P\{\text{choose "1" |"0" was sent}\} = P\{y(nT) > A |"0" \text{ was sent}\}
\]

\[
= \int_{A}^{\infty} f(y(nT)|"0" \text{ was sent})dy.
\]
So, the average probability of a bit error (BER) is given by

\[
P\{\text{bit error}\} = P\{\{\text{bit error and "0" was sent}\} \cup \{\text{bit error and "1" was sent}\}\} \\
= P\{\text{bit error and "0" was sent}\} + P\{\text{bit error and "1" was sent}\} \\
= P\{\text{choose "1" and "0" was sent}\} + P\{\text{choose "0" and "1" was sent}\} \\
= P\{\text{choose "1" | "0" was sent}\} P\{"0" was sent\} \\
+ P\{\text{choose "0" | "1" was sent}\} P\{"1" was sent\} \\
= p \int_{-\infty}^{\infty} f(y(nT)|"0" was sent)dy + (1-p) \int_{-\infty}^{A} f(y(nT)|"1" was sent)dy
\]

where \( p = P\{"0" was sent\} \) and \( 1-p = P\{"1" was sent\} \).

The decision regions are separated by the threshold \( A \). See the next figure, where the threshold \( A \) is assumed to be 0 because the bits occur with equal probability, i.e. \( p = 0.5 \) (we will discuss this issue next).
\( f(y|\text{"0" was sent}) \quad f(y|\text{"1" was sent}) \)
The question now is: How do we choose \( A \) optimally?

To find the optimal value of \( A \), we minimize the probability of error with respect to it, i.e.,

\[
\frac{d}{dA} P\{\text{bit error}\} = \frac{d}{dA} \left[ p \int_{-\infty}^{A} f(y(nT)|0\text{ was sent}) \, dy + (1 - p) \int_{-\infty}^{A} f(y(nT)|1\text{ was sent}) \, dy \right]
\]

\[
= -p f(A|0\text{ sent}) + (1 - p) f(A|1\text{ sent}) = 0
\]

or

\[
\frac{d}{dA} P\{\text{bit error}\} = \frac{1}{\sqrt{2\pi} \sigma_N} \left[ -pe^{-(A-G_2)^2/2\sigma_N^2} + (1 - p)e^{-(A-G_1)^2/2\sigma_N^2} \right] = 0
\]

\[
\Rightarrow \frac{1 - p}{p} = \frac{e^{-(A-G_2)^2/2\sigma_N^2}}{e^{-(A-G_1)^2/2\sigma_N^2}} = e^{-\frac{1}{2\sigma_N^2}[(A-G_2)^2 - (A-G_1)^2]}
\]

or

\[
\ln \left[ \frac{1 - p}{p} \right] = -\frac{1}{2\sigma_N^2} [(A-G_2)^2 - (A-G_1)^2]
\]

or

\[
\ln \left[ \frac{p}{1 - p} \right] = \frac{1}{2\sigma_N^2} \left[ A^2 - 2AG_2 + G_2^2 - A^2 + 2AG_1 - G_1^2 \right] = \frac{1}{2\sigma_N^2} \left[ G_2^2 - G_1^2 - 2A(G_2 - G_1) \right]
\]
The optimal value of $A$ can now be found by solving the following equation for $A$:

$$\sigma_N^2 \ln \left[ \frac{p}{1-p} \right] = \frac{G_2^2 - G_1^2}{2} - A(G_2 - G_1) = \frac{(G_2 - G_1)(G_2 + G_1)}{2} - A(G_2 - G_1).$$

Namely, $A_{opt} = \frac{G_1 + G_2}{2} + \left( \frac{\sigma_N^2}{G_1 - G_2} \right) \ln \left[ \frac{p}{1-p} \right]$.

If both symbols are transmitted with equal probability, i.e. $p = \frac{1}{2}$ then $A_{opt} = \frac{G_1 + G_2}{2}$, namely, the arithmetic average of the means at the output of $h(t)$ at $t = nT$. When the two symbols are not transmitted with equal probability, the optimal threshold $A_{opt}$ will shift to the right or to the left, depending on which symbol occurs with higher probability.
In the case of equal probability of occurrence, the bit error rate (BER) is given by

\[
P\{\text{bit error}\} = \left[ \frac{1}{2} \int_{A_{opt}}^{\infty} f(y(nT) \mid "0" \text{sent}) \, dy + \frac{1}{2} \int_{-\infty}^{A_{opt}} f(y(nT) \mid "1" \text{sent}) \, dy \right]
\]

\[
= \frac{1}{2\sqrt{2\pi}\sigma_N} \left[ \int_{G_1+G_2}^{\infty} e^{-(y-G_2)^2/2\sigma_N^2} \, dy + \int_{-\infty}^{\infty} e^{-(y-G_1)^2/2\sigma_N^2} \, dy \right]
\]

\[
= \frac{1}{2\sqrt{2\pi}\sigma_N} \left[ \sigma_N \int_{G_1-G_2}^{\infty} e^{-u^2/2} \, du + \sigma_N \int_{-\infty}^{\infty} e^{-v^2/2} \, dv \right], \quad u = \frac{y-G_2}{\sigma_N}, \quad v = -\frac{y-G_1}{\sigma_N}
\]

\[
= \frac{1}{\sqrt{2\pi}} \int_{\frac{G_1-G_2}{2\sigma_N}}^{\infty} e^{-u^2/2} \, du = Q\left(\frac{G_1-G_2}{2\sigma_N}\right),
\]

where \( Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-\frac{u^2}{2}} \, du, \quad x \geq 0, \) is the area under the tail of the Gaussian pdf with zero mean and unit variance, i.e. the area under the tail of \( f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \forall x. \)
It should be clear from the previous derivation that both $G_1$ and $G_2$ depend on $h(t)$, the impulse response of the receiver filter. Also, $Q\left(\frac{G_1 - G_2}{2\sigma_N}\right)$ decreases as $\frac{G_1 - G_2}{2\sigma_N}$ increases, i.e., the average probability of error, $P\{\text{bit error}\}$, decreases as the separation between $G_1$ and $G_2$ increases.

Let’s now find the $h(t)$ that will result in the minimum probability of bit error. To do this, consider the following optimization problem:

Maximize over all possible $h(t)$ the square of the argument of the Q function, i.e.

$$\max_{h(t)} \left\{ \left(\frac{G_1 - G_2}{2\sigma_N}\right)^2 \right\}.$$
\[
\max_{h(t)} \left\{ \left( \frac{G_1 - G_2}{2\sigma_N} \right)^2 \right\} = \max_{h(t)} \left\{ \frac{(G_1 - G_2)^2}{\sigma_N} \right\} \\
= \max_{h(t)} \left\{ \frac{N_0}{2} \int_{-\infty}^{\infty} \left[ \int h(\tau)g_1(nT-\tau)d\tau - \int h(\tau)g_2(nT-\tau)d\tau \right]^2 \right\} \\
= \max_{h(t)} \left\{ \frac{N_0}{2} \int_{-\infty}^{\infty} |H(f)|^2 df \right\} \\
= \max_{h(t)} \left\{ \frac{N_0}{2} \int_{-\infty}^{\infty} |H(f)|^2 df \right\} \\
= \max_{h(t)} \left\{ \left( h(t) \ast [g_1(t) - g_2(t)] \right)^2 \right\}.
\]
But,
\[
\left( h(t) * [g_1(t) - g_2(t)] \right)^2 = \left( \int_{-\infty}^{\infty} H(f) [G_1(f) - G_2(f)] e^{j2\pi ft} df \right)^2, \ t = nT
\]

Using the Schwarz inequality, we get
\[
\left( h(t) * [g_1(t) - g_2(t)] \right)^2 \leq \int_{-\infty}^{\infty} \left| [G_1(f) - G_2(f)] e^{j2\pi ft} \right|^2 df \cdot \int_{-\infty}^{\infty} |H(f)|^2 df, \ t = nT,
\]
\[
= \int_{-\infty}^{\infty} |G_1(f) - G_2(f)|^2 df \cdot \int_{-\infty}^{\infty} |H(f)|^2 df, \ t = nT
\]

The maximum is achieved when equality occurs. When this is the case,
\[
H_{opt}(f) = k \left( [G_1(f) - G_2(f)] * e^{j2\pi f nT} \right)^*
\]
\[
= k [G_1(f) - G_2(f)]^* e^{-j2\pi f nT}.
\]

Since \( k \) is arbitrary, we let \( k = 1 \). Thus,
\[
\max_{h(t)} \left\{ \left( \frac{G_1 - G_2}{\sigma_N} \right)^2 \right\} = \frac{2}{N_0} \int_{-\infty}^{\infty} |G_1(f) - G_2(f)|^2 df.
\]
For \( k = 1 \),
\[ H_{opt}(f) = \left[ G_1(f) - G_2(f) \right]^* e^{-j2\pi fnT} \]
and
\[
h_{opt}(t) = \int_{-\infty}^{\infty} H_{opt}(f)e^{j2\pi ft} df = \int_{-\infty}^{\infty} \left[ G_1^*(f) - G_2^*(f) \right] e^{-j2\pi f(nT-t)} df
\]
\[
= \left( \int_{-\infty}^{\infty} \left[ G_1(f) - G_2(f) \right] e^{j2\pi f(nT-t)} df \right)^*
\]
\[
= \left( g_1(nT-t) - g_2(nT-t) \right)^*
\]
\[
= g_1^*(nT-t) - g_2^*(nT-t)
\]
\[
= g_1(nT-t) - g_2(nT-t), \text{ for real } g_1(t), g_2(t).
\]

Assuming \( p = \frac{1}{2} \), \( P\{\text{error}\} \) is minimum when \( \left( \frac{G_1 - G_2}{\sigma_N} \right)^2 \) is maximum or when
\[
h(t) = g_1(nT-t) - g_2(nT-t), \text{ i.e. } h(t) \text{ matches the input pulses } g_1(t) \text{ and } g_2(t).
By Parseval’s theorem,

\[
\max_{h(t)} \left\{ \left( \frac{G_1 - G_2}{\sigma_N} \right)^2 \right\} = \frac{2}{N_0} \int_{-\infty}^{\infty} |G_1(f) - G_2(f)|^2 df = \frac{2}{N_0} \int_{-\infty}^{\infty} \left[ g_1(t) - g_2(t) \right]^2 dt,
\]

which implies that

\[
\left( \frac{G_1 - G_2}{2\sigma_N} \right)_{\text{max}} = \frac{1}{2} \left[ \frac{2}{N_0} \int_{-\infty}^{\infty} \left[ g_1(t) - g_2(t) \right]^2 dt \right]^{1/2} = \frac{1}{2} \left[ \frac{2}{N_0} \int_{-\infty}^{\infty} \left| G_1(f) - G_2(f) \right|^2 df \right]^{1/2}
\]

\[
= \frac{1}{\sqrt{2N_0}} \left( \int_{-\infty}^{\infty} \left[ g_1(t) - g_2(t) \right]^2 dt \right) = \frac{1}{\sqrt{2N_0}} \left( \int_{-\infty}^{\infty} \left| G_1(f) - G_2(f) \right|^2 df \right)
\]

and

\[
P\{\text{error}\}_{\text{min}} = Q \left( \sqrt{ \frac{\int_{-\infty}^{\infty} \left[ g_1(t) - g_2(t) \right]^2 dt}{2N_0} } \right) = Q \left( \sqrt{ \frac{\int_{-\infty}^{\infty} \left| G_1(f) - G_2(f) \right|^2 df}{2N_0} } \right).
\]

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Example: Compute the minimum error probability (BER) for the following on-off keying (transmit a pulse when a logical 1 occurs and transmit nothing when a logical 0 occurs):

Here, \( g_1(t) = A \cdot tr \left( \frac{t - \frac{T}{2}}{\frac{T}{2}} \right) \) and \( g_2(t) = 0, \ 0 \leq t \leq T. \)
Now,
\[
\int_{-\infty}^{\infty} \left[ g_1(t) - g_2(t) \right]^2 dt = \int_0^T \left[ A t r \left( \frac{t - T/2}{T} \right) - 0 \right]^2 dt
\]
\[
= A^2 \int_0^T t r^2 \left( \frac{t - T/2}{T} \right) dt
\]
\[
= A^2 \left[ \int_0^{T/2} \left( \frac{2}{T} t \right)^2 dt + \int_{T/2}^T \left[ -\frac{2}{T} (t - T) \right]^2 dt \right]
\]
\[
= A^2 \left[ 2 \int_0^{T/2} \frac{4}{T^2} t^2 dt \right] = \frac{8A^2}{3T^2} T^3 \left|_0^{T/2} \right.
\]
\[
= \frac{A^2T}{3}
\]

and the minimum bit error rate (BER) is equal to
\[
P\{ \text{error occurs} \}_{\text{min}} = Q \left( \sqrt{\frac{A^2T}{6N_0}} \right).
\]
Assuming the bit sequence is random, then the average bit energy is given by

\[ E_{b,av} = E_{\text{\textquoteleft\textprime}1} \left( \frac{1}{2} \right) + E_{\text{\textquoteleft\textprime}0} \left( \frac{1}{2} \right) = \frac{A^2 T}{3} \left( \frac{1}{2} \right) + 0 \left( \frac{1}{2} \right) = \frac{A^2 T}{6} \]

and the BER can be rewritten as

\[ BER_{\text{min}} = Q \left( \sqrt{\frac{E_{b,av}}{N_0}} \right) \]

The following plot shows the performance of the previous communication system in AWGN (this is the same as on-off keying or OOK).
Baseband signal-space Analysis

Our goal now is to formulate the different detection strategies in a more intuitive fashion. We do this by giving the modulated time signals a geometric interpretation.

Let $S$ be a $K$-dimensional signal space and $\{\phi_i(t)\}_{i=1}^K$ be a basis for this space. Suppose further that the basis functions are orthonormal, i.e.

$$\int_{t_0}^{t_0 + T_s} \phi_i(t)\phi_j^*(t)dt = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases},$$

where $T_s$ is a time interval yet to be determined, then $\{\phi_i(t)\}_{i=1}^K$ is an orthonormal set.

Let $s(t) \triangleq \sum_{j=1}^K s_j \phi_j(t)$, i.e., $s(t) \in S$. 
Now, for $i = 1, \ldots, K$

$$
\int_{t_0}^{t_0+T_s} s(t)\phi_i^*(t)dt = \int_{t_0}^{t_0+T_s} \left( \sum_{j=1}^{K} s_j \phi_j(t) \right) \phi_i^*(t)dt
= \sum_{j=1}^{K} s_j \int_{t_0}^{t_0+T_s} \phi_j(t)\phi_i^*(t)dt = s_i, i = 1, \ldots, K.
$$

Furthermore, the energy of $s(t)$ is given by

$$
E = \int_{t_0}^{t_0+T_s} \left| s(t) \right|^2 dt = \int_{t_0}^{t_0+T_s} s(t)s^*(t)dt = \int_{t_0}^{t_0+T_s} \left( \sum_{i=1}^{K} s_i \phi_i(t) \right)\left( \sum_{j=1}^{K} s_j \phi_j(t) \right)^* dt
= \sum_{i=1}^{K} \sum_{j=1}^{K} s_i s_j^* \int_{t_0}^{t_0+T_s} \phi_i(t)\phi_j^*(t)dt
= \sum_{i=1}^{K} s_i s_i^* = \sum_{i=1}^{K} \left| s_i \right|^2.
$$
Let the coefficients $s_i$, $i = 1, \ldots, K$ be expressed as a vector $\vec{s} \triangleq [s_1, s_2 \ldots s_K]^T$, then

$$E = \vec{s}^T \vec{s}^* = \left[ s_1 \ldots s_K \right] \begin{bmatrix} s_1^* \\ \vdots \\ s_K^* \end{bmatrix} = \sum_{i=1}^{K} |s_i|^2,$$

i.e., $E$ is the inner (dot) product of $\vec{s}$ with itself.

Let $\{s_1(t), \ldots s_M(t)\}$ be a set of signals we want to use in a communication system. If this set is defined on the interval $(t_0, t_0 + T_s)$, where $T_s$ is the maximum signal duration, then an orthonormal basis can be constructed as follows:

1. Let $g_1(t) = s_1(t)$ and $\phi_1(t) = g_1(t)/\|g_1(t)\| = g_1(t)/\sqrt{E_1} \Rightarrow s_1(t) = \sqrt{E_1}\phi_1(t),$

where $\|g_1(t)\| \triangleq \left[ \int_{t_0}^{t_0 + T_s} g_1(t)g_1^*(t) dt \right]^{\frac{1}{2}} = \left[ \int_{t_0}^{t_0 + T_s} |g_1(t)|^2 dt \right]^{\frac{1}{2}} = \sqrt{E_1}.$
2. Let \( g_2(t) = s_2(t) - \langle s_2(t), \phi_1(t) \rangle \phi_1(t) \) and \( \phi_2(t) = g_2(t)/\|g_2(t)\| \),

where \( \langle u(t), v(t) \rangle \triangleq \int_{t_0}^{t_0+T} u(t)v^*(t)dt. \)

3. Let \( g_3(t) = s_3(t) - \langle s_3(t), \phi_1(t) \rangle \phi_1(t) - \langle s_3(t), \phi_2(t) \rangle \phi_2(t) \), \( \phi_3(t) = \frac{g_3(t)}{\|g_3(t)\|}. \)

\[ \vdots \]

K. Let \( g_K(t) = s_K(t) - \sum_{i=1}^{K-1} \langle s_K(t), \phi_i(t) \rangle \phi_i(t) \), \( \phi_K = \frac{g_K(t)}{\|g_K(t)\|}, \|g_k(t)\| \neq 0 \) and \( K \leq M. \)

Note that \( \langle s_k(t), \phi_i(t) \rangle \) can be interpreted as the projection of \( s_k(t) \) onto \( \phi_i(t) \). The set of basis function \( \{\phi_i(t)\}_{i=1}^K \) forms an orthonormal set. The procedure outlined above is known as the Gram-Schmidt orthogonalization procedure.
Remark 1: The set of signals \( \{s_i(t)\}_{i=1}^{M} \) is a linearly independent set iff \( K = M \).

Remark 2: The signals \( s_1(t), \ldots s_M(t) \) are not linearly independent if \( K < M \) and \( g_i(t) = 0 \), \( K < i \leq M \).

Example: Consider the signals \( s_i(t), i = 1, 2, 3, 4 \) (\( M = 4 \)) described by

\[
\begin{align*}
s_1(t) & \quad 1 \quad 1 \quad 2 \quad 3 \quad 4 \\
0 & \quad 1 & \quad 2 & \quad 3 & \quad 4 & \quad t
\end{align*}
\]

\[
\begin{align*}
s_2(t) & \quad 1 \quad 1 \quad 2 \quad 3 \\
0 & \quad 1 & \quad 2 & \quad 3 & \quad 4 & \quad t
\end{align*}
\]

\[
\begin{align*}
s_3(t) & \quad 1 \quad 2 \quad 3 \quad 4 \\
0 & \quad 1 & \quad 2 & \quad 3 & \quad 4 & \quad t
\end{align*}
\]

\[
\begin{align*}
s_4(t) & \quad 1 \quad 2 \quad 3 \quad 4 \\
0 & \quad 1 & \quad 2 & \quad 3 & \quad 4 & \quad t
\end{align*}
\]

In this case \( T_s = 3 \) seconds. Let us now construct an orthonormal basis for this set of signals.

1. \( g_1(t) = s_1(t) \)

\[
\|g_1(t)\| = \left( \left\langle g_1(t), g_1(t) \right\rangle \right)^{\frac{1}{2}} = \left( \int_{0}^{3} |g_1(t)|^2 \, dt \right)^{\frac{1}{2}} = \left( \int_{0}^{1} 1 \, dt \right)^{\frac{1}{2}} = 1 = \sqrt{E_1}
\]

\[
\phi_1(t) = \frac{g_1(t)}{\|g_1(t)\|} = \frac{g_1(t)}{1} = g_1(t) = s_1(t) \Rightarrow s_1(t) = \phi_1(t)
\]
2. \( g_2(t) = s_2(t) - \langle s_2(t), \phi_1(t) \rangle \phi_1(t) \)

\[
\langle s_2(t), \phi_1(t) \rangle = \int_0^1 s_2(t) \phi_1^*(t) \, dt = \int_0^1 1 \, dt = 1 \Rightarrow g_2(t) = s_2(t) - \phi_1(t)
\]

or \( g_2(t) = s_2(t) - s_1(t) \). Now,

\[
\|g_2(t)\| = \left( \int_0^1 |g_2(t)|^2 \, dt \right)^{\frac{1}{2}} = \left( \int_0^1 [s_2(t) - s_1(t)]^2 \, dt \right)^{\frac{1}{2}} = \left( \int_1^2 1 \, dt \right)^{\frac{1}{2}} = 1 = \sqrt{E_2}
\]

\[
\phi_2(t) = \frac{g_2(t)}{\|g_2(t)\|} = \frac{s_2(t) - \phi_1(t)}{1} \Rightarrow s_2(t) = \phi_1(t) + \phi_2(t)
\]

3. \( g_3(t) = s_3(t) - \langle s_3(t), \phi_1(t) \rangle \phi_1(t) - \langle s_3(t), \phi_2(t) \rangle \phi_2(t) \)
\[ \langle s_3(t), \phi_1(t) \rangle = \int_{0}^{3} s_3(t)\phi_1^*(t) \, dt = \int_{0}^{3} s_3(t)s_1(t) \, dt = 0 \]

\[ \langle s_3(t), \phi_2(t) \rangle = \int_{0}^{3} s_3(t)\phi_2^*(t) \, dt = \int_{0}^{3} s_3(t)[s_2(t) - s_1(t)] \, dt = \int_{1}^{2} 1 \, dt = 1 \]

\[ g_3(t) = s_3(t) - \phi_2(t) = s_3(t) - s_2(t) + \phi_1(t) = s_3(t) - s_2(t) + s_1(t) \]

\[ \|g_3(t)\| = \left( \int_{0}^{3} |g_3(t)|^2 \, dt \right)^{\frac{1}{2}} = \left( \int_{0}^{3} [s_3(t) - s_2(t) + s_1(t)]^2 \, dt \right)^{\frac{1}{2}} = \left( \int_{2}^{3} 1 \, dt \right)^{\frac{1}{2}} = 1 = \sqrt{E_3} \]

\[ \phi_3(t) = \frac{g_3(t)}{\|g_3(t)\|} = \frac{s_3(t) - \phi_2(t)}{1} \implies s_3(t) = \phi_2(t) + \phi_3(t) \]

4. \[ g_4(t) = s_4(t) - \langle s_4(t), \phi_1(t) \rangle \phi_1(t) - \langle s_4(t), \phi_2(t) \rangle \phi_2(t) - \langle s_4(t), \phi_3(t) \rangle \phi_3(t) \]

\[ \langle s_4(t), \phi_1(t) \rangle = \int_{0}^{3} s_4(t)\phi_1^*(t) \, dt = \int_{0}^{1} 1 \, dt = 1 \]

\[ \langle s_4(t), \phi_2(t) \rangle = \int_{0}^{3} s_4(t)\phi_2^*(t) \, dt = \int_{1}^{2} 1 \, dt = 1 \]
\[ \langle s_4(t), \phi_3(t) \rangle = \int_0^3 s_4(t)\phi_3^*(t)\,dt = \int_0^2 1\,dt = 1 \]

\[ \therefore g_4(t) = s_4(t) - \phi_1(t) - \phi_2(t) - \phi_3(t) = s_4(t) - [\phi_1(t) + \phi_2(t) + \phi_3(t)] \]

But, \( \phi_i(t), i = 1, 2, 3 \) are described by

\[
\begin{align*}
\phi_1(t) &= \begin{cases} 1 & 0 < t < 1 \\ 0 & \text{otherwise} \end{cases} \\
\phi_2(t) &= \begin{cases} 1 & 1 < t < 2 \\ 0 & \text{otherwise} \end{cases} \\
\phi_3(t) &= \begin{cases} 1 & 2 < t < 3 \\ 0 & \text{otherwise} \end{cases}
\end{align*}
\]

\[ g_4(t) = 0 \Rightarrow s_4(t) = \phi_1(t) + \phi_2(t) + \phi_3(t) \]

Hence,

\[ s_1(t) = \phi_1(t) \]
\[ s_2(t) = \phi_1(t) + \phi_2(t) \]
\[ s_3(t) = \phi_2(t) + \phi_3(t) \]
Clearly, $\{s_i(t)\}_{i=1}^4$ is defined on the 3-dimensional Euclidean space represented by the coordinates $\phi_1, \phi_2$ and $\phi_3$.

Let the signal arriving at the receiver be described by $x(t) = s_i(t) + W(t)$, $i = 1, \ldots, M$, where $s_i(t)$ is the transmitted signal and $W(t)$ is WGN with zero mean and power spectral density $S_w(f) = \frac{N_0}{2} \text{Watts/Hz}$, $\forall f$. Let $\{\phi_j(t)\}_{j=1}^K$, $K \leq M$ be an orthonormal basis for the signal space $S$, i.e., $s_i(t) = \sum_{j=1}^K s_j \phi_j(t), i = 1, \ldots, M$. 
Consider a coherent correlator receiver and the observed output at the k\textsuperscript{th} correlator, i.e.,

\[ \int_{0}^{T_s} x(t) e^{j2\pi f_0 t} \, dt \]

\[ X_k \]

where \( x(t) \) is the input signal, \( f_0 \) is the carrier frequency, and \( T_s \) is the symbol interval. The diagram illustrates the coherent correlator receiver with multiple correlators, each operating at different times \( t = T_s \).
Let $s_i(t)$ or symbol $m_i$ be transmitted through the channel, then the output of the $k^{th}$ correlator is given by

$$
X_k | m_i = \int_0^{T_s} x(t)\phi_k^*(t)dt = \int_0^{T_s} [s_i(t) + W(t)]\phi_k^*(t)dt = \int_0^{T_s} \left[ \sum_{j=1}^{K} s_{ij}\phi_j(t) + W(t) \right] \phi_k^*(t)dt
$$

$$
= \sum_{j=1}^{K} s_{ij} \int_0^{T_s} \phi_j(t)\phi_k^*(t)dt + \int_0^{T_s} W(t)\phi_k^*(t)dt = s_{ik} + W_k, \quad k = 1, \ldots, K,
$$

where, $s_{ik} = \int_0^{T_s} s_i(t)\phi_k^*(t)dt$ and $W_k = \int_0^{T_s} W(t)\phi_k^*(t)dt$.

Define a new r.p. $x'(t)$ by $x'(t) \overset{\Delta}{=} x(t) - \sum_{k=1}^{K} X_k \phi_k(t) = x(t) - \bar{X}^T \Phi(t)$, where

$$
\bar{X} = \begin{bmatrix} X_1 & \ldots & X_K \end{bmatrix}^T
$$

is the projection of $x(t)$ onto the signal space $S$ and

$$
\Phi(t) = \begin{bmatrix} \phi_1(t) & \ldots & \phi_K(t) \end{bmatrix}^T.
$$
Hence,

\[ x'(t) = s_i(t) + W(t) - \sum_{k=1}^{K} X_k \phi_k(t) = s_i(t) + W(t) - \sum_{k=1}^{K} \left[ s_{ik} + W_k \right] \phi_k(t) \]

\[ = \sum_{k=1}^{K} s_{ik} \phi_k(t) + W(t) - \sum_{k=1}^{K} s_{ik} \phi_k(t) - \sum_{k=1}^{K} W_k \phi_k(t) = W(t) - \sum_{k=1}^{K} W_k \phi_k(t) \]

\[ = W(t) - \begin{bmatrix} \phi_1(t) & \cdots & \phi_K(t) \end{bmatrix} \begin{bmatrix} W_1 \\ \vdots \\ W_K \end{bmatrix} \overset{\triangle}{=} W(t) - \Phi^T(t) \tilde{W} = \sum_{k=K+1}^{\infty} W_k \phi_k(t) = W'(t), \]

where, \( \tilde{W} = [W_1, \ldots, W_K]^T \) is the projection of the noise onto the space \( S \) (signal space) and \( W'(t) \) is the part of the noise \( W(t) \) that does not lie on the signal space \( S \). Therefore,

\[ x(t) = \sum_{k=1}^{K} X_k \phi_k(t) + W'(t) , \] which means that we must only worry about the part of the noise that lies on the signal space, namely, the part of the noise which is not in the signal space does not affect the output of the correlators.
Define $W(t) \triangleq W_r(t) + W_p(t)$, where $W_r(t) \triangleq \sum_{k=1}^{K} W_k \phi_k(t)$.

Then $x(t) = s_i(t) + W_r(t) + W_p(t)$, where $W_p(t) = W'(t)$

Now, if $x(t)$ is a Gaussian r.p., then $X_k \mid m_i$ is a Gaussian r.v. with mean

$$\bar{X}_k \mid m_i = E \{ X_k \mid m_i \} = E \left\{ \int_0^{T_s} x(t) \phi_k^*(t) dt \mid m_i \right\} = E \left\{ \int_0^{T_s} \left[ \sum_{j=1}^{K} s_{ij} \phi_j(t) + W(t) \right] \phi_k^*(t) dt \mid m_i \right\}$$

$$= E \left\{ \sum_{j=1}^{K} s_{ij} \int_0^{T_s} \phi_j(t) \phi_k^*(t) dt + \int_0^{T_s} W(t) \phi_k^*(t) dt \mid m_i \right\}$$

$$= E \left\{ \sum_{j=1}^{K} s_{ij} \int_0^{T_s} \phi_j(t) \phi_k^*(t) dt \mid m_i \right\} + E \left\{ \int_0^{T_s} W(t) \phi_k^*(t) dt \mid m_i \right\}$$

$$= s_{ik} + \int_0^{T_s} \underbrace{E \{ W(t) \phi_k^*(t) \}}_{0} dt = s_{ik}$$

and variance
\[
\sigma^2_{X_k|m_i} = E \left\{ |X_k - s_{ik}|^2 \mid m_i \right\} = E \left\{ |W_k|^2 \mid m_i \right\} = E \left\{ \int_0^{T_s} \int_0^{T_s} W(t)W^*(\tau)\phi_k(\tau)\phi_k^*(t)dtd\tau \right\}
\]
\[
= \frac{N_0}{2} \int_0^{T_s} \int_0^{T_s} \delta(t-\tau)\phi_k^*(t)\phi_k(\tau)dtd\tau
\]
\[
= \frac{N_0}{2} \int_0^{T_s} \phi_k^*(\tau)\phi_k(\tau)d\tau = \frac{N_0}{2}, \quad k = 1, \ldots, K.
\]

Also, for \( j \neq k \),

\[
E \left\{ (X_j - s_{ij})(X_k - s_{ik})^* \mid m_i \right\} = E \left\{ W_jW_k^* \mid m_i \right\} = E \left\{ \int_0^{T_s} \int_0^{T_s} W(t)\phi_j^*(t) \cdot W^*(\tau)\phi_k(\tau)dtd\tau \right\}
\]
\[
= \frac{N_0}{2} \int_0^{T_s} \int_0^{T_s} \delta(t-\tau)\phi_k(\tau)\phi_j^*(t)dtd\tau = \frac{N_0}{2} \int_0^{T_s} \phi_k(\tau)\phi_j^*(\tau)d\tau = 0
\]
This means that the $X_k'$s are mutually uncorrelated $\Rightarrow X_k'$s are statistically independent because they are Gaussian. Hence, the joint density function of $\mathbf{X} = [X_1 \ldots X_K]^T$, given that message $m_i$ has been transmitted is given by

$$f_{\mathbf{x}} (x|m_i) = f_{\mathbf{x}} (x_1, \ldots, x_K | m_i) = \prod_{k=1}^{K} f_{X_k} (x_k | m_i), \ i = 1, \ldots, M$$

$$= \prod_{k=1}^{K} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(x_k-s_{ik})^2}{2N_0}} = \frac{1}{(\pi N_0)^{K/2}} e^{-\frac{\sum_{k=1}^{K} (x_k-s_{ik})^2}{2N_0}}.$$  

Define the Euclidean distance between vectors $\mathbf{u}$ and $\mathbf{v}$ by

$$\|\mathbf{u} - \mathbf{v}\| = \left[ (u_1 - v_1)^2 + \cdots + (u_K - v_K)^2 \right]^{1/2} = \left[ \sum_{k=1}^{K} (u_k - v_k)^2 \right]^{1/2}, \text{ where } \mathbf{u} = [u_1 \ldots u_K]^T$$

and $\mathbf{v} = [v_1 \ldots v_K]^T$. Then

$$f_{\mathbf{x}} (\mathbf{x}|m_i) = \frac{1}{(\pi N_0)^{K/2}} \exp \left\{ -\frac{1}{N_0} \|\mathbf{x} - \mathbf{s}_i\|^2 \right\}, \ i = 1, \ldots, M \text{ where } \mathbf{s}_i = [s_{i1} \ s_{i2} \ldots \ s_{iK}]^T.$$
Let $s_i$ be transmitted and $\mathcal{X}$ be the observation vector of the sampled values of the outputs of the $K$ correlators, then
\[ \mathcal{X} = [X_1 \ldots X_K]^T = s_i + \mathcal{W}, \quad \mathcal{W} = [W_1 \ldots W_K]^T. \]

**Decision strategy:** Given the observation vector $\mathcal{X} = \mathcal{X}$, choose the symbol $m_k$ so that the probability of making a decision error is minimum.

Let symbol (signal) $m_i$ be sent through the channel and let $P_e(m_i|\mathcal{X})$ denote the conditional probability of making a decision error given that $\mathcal{X}$ is observed, then
\[ P_e(m_i|\mathcal{X}) \equiv P\{\text{select } m_k, k \neq i|\mathcal{X}\} = 1 - P\{\text{select } m_i|\mathcal{X}\} \]

**Optimum Decision Rule:** $P_e(m_i|\mathcal{X})$ is minimum whenever $P\{\text{select } m_i|\mathcal{X}\}$ is maximum. Equivalently, choose $m_i$ if
\[ P\{\text{select } m_i|\mathcal{X}\} \geq P\{\text{select } m_k|\mathcal{X}\}, \forall k \neq i, k = 1, \ldots, M. \]

This is known as the Maximum A Posteriori (MAP) probability. Equivalently, applying Bayes’ rule, yields:

Choose symbol $m_i$ if \[ \frac{p_k f_X(x|m_k)}{f_X(\mathcal{X})} \text{ is maximum for } k = i, k = 1, \ldots, M, \]
where \( p_k \) is the a priori probability of occurrence of the symbol \( m_k \), \( f_x(x|m_k) \) is the likelihood function that results when \( m_k \) is transmitted, and \( f_x(x) \) is the unconditional p.d.f. of \( X \).

The equivalent rule comes from the fact that, in the limit, Bayes’ rule, as applied to a continuous r.v. is given by

\[
P\{A|X = x\} = \frac{f_x(x|A)P\{A\}}{f_x(x)},
\]

where \( A \) is an event of selecting symbol \( m_k \).

The distribution of \( X \) is independent of the transmitted signal. Therefore, if \( p_k = p \), i.e., all symbols are transmitted with equal probability, then the optimum decision rule can be stated as

Choose \( m_i \) if \( f_x(x|m_k) \) is maximum for \( k = i, k = 1, \ldots, M \).

This is the **maximum likelihood** decision rule and is based on Bayesian statistics.
Finally, since the likelihood function is non-negative because it is a probability density function, we can restate the optimum decision rule as

choose \( m_i \) if \( \ln \left[ f_{\tilde{x}} \left( \tilde{x} | m_k \right) \right] \) is maximum for \( k = i, k = 1, \ldots, M \), since \( \ln (\bullet) \) is a monotonically increasing function of the argument.

**Remark:** The maximum likelihood decision rule differs from the MAP decision rule in that it assumes equally likely message symbols.

For an AWGN channel, the conditional pdf of the observation vector \( \tilde{x} \) given that symbol \( m_k \) was transmitted is described by

\[
f_{\tilde{x}} \left( \tilde{x} | m_k \right) = \frac{1}{\left( \pi N_0 \right)^{K/2}} e^{- \frac{1}{N_0} \| \tilde{x} - s_k \|^2}, \quad k = 1, \ldots, M.
\]

Hence,

\[
\ln \left[ f_{\tilde{x}} \left( \tilde{x} | m_k \right) \right] = - \frac{K}{2} \ln [\pi N_0] - \frac{1}{N_0} \| \tilde{x} - s_k \|^2, \quad k = 1, \ldots, M.
\]
But,

\[
\max_k \left\{ \ln \left[ f_\mathbf{x} \left( \mathbf{z} \mid m_k \right) \right] \right\} = \max_k \left\{ -\frac{K}{2} \ln [\pi N_0] - \frac{1}{N_0} \| \mathbf{z} - \mathbf{s}_k \|^2 \right\} \\
= \max_k \left\{ -\frac{1}{N_0} \| \mathbf{z} - \mathbf{s}_k \|^2 \right\} = \min_k \left\{ \frac{1}{N_0} \| \mathbf{z} - \mathbf{s}_k \|^2 \right\},
\]

\[
= \min_k \left\{ \| \mathbf{z} - \mathbf{s}_k \|^2 \right\} = \min_k \left\{ \| \mathbf{z} - \mathbf{s}_k \| \right\}
\]

since multiplication by the positive constant \( \frac{1}{N_0} \) does not change the location of the minimum.

Geometrically speaking, if we partition the \( K \)-dimensional signal space into \( M \) regions, \( R_1, \ldots, R_M \), then the decision rule can be reformulated as follows:

\( \mathbf{x} \) lies inside \( R_i \) if \( \ln \left[ f_\mathbf{x} \left( \mathbf{z} \mid m_k \right) \right] \) is maximum for \( k = i, k = 1, \ldots, M \).

Therefore,

\( \mathbf{z} \) lies inside \( R_i \) if the Euclidean distance \( \| \mathbf{z} - \mathbf{s}_k \| \) is minimum for \( k = i, k = 1, \ldots, M \),

i.e., choose \( m_i \) if the distance between \( \mathbf{z} \) and \( \mathbf{s}_i \) is minimum.
Error Performance of MAP Receivers

If symbol $m_i$ (signal vector $s_i$) is transmitted and $x$ does not lie in $R_i$, then an error occurs. Therefore, the average probability of symbol error (SER) is

$$P_e = \sum_{i=1}^{M} P\{X \text{ does not lie in } R_i \text{ and } m_i \text{ was sent}\}$$

$$= \sum_{i=1}^{M} P\{X \text{ does not lie in } R_i | m_i \text{ was sent}\}P\{m_i \text{ was sent}\}$$

If $P\{m_i \text{ was sent}\} = \frac{1}{M}$, $\forall i, i = 1, \ldots, M$, then

$$P_e = \frac{1}{M} \sum_{i=1}^{M} P\{X \text{ does not lie in } R_i | m_i \text{ sent}\}$$

or

$$P_e = 1 - \frac{1}{M} \sum_{i=1}^{M} P\{X \text{ lies in } R_i | m_i \text{ sent}\},$$

where $P\{X \text{ lies in } R_i | m_i \text{ sent}\} = \int_{R_i} f_X(x|m_i) dx$. 

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Example: Let $m(t)$ be a binary signal transmitted over an AWGN channel. Let $m(t)$ be represented by a bipolar nonreturn to zero waveform with amplitude $A$. Then

$$s_1(t) = \begin{cases} A, & 0 \leq t \leq T_b \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad s_2(t) = \begin{cases} -A, & 0 \leq t \leq T_b \\ 0, & \text{otherwise} \end{cases}.$$ 

Let us apply the Gram-Schmidt procedure.

Let $g_1(t) = s_1(t)$, then

$$\|g_1(t)\| = \left( \int_0^{T_b} A^2 dt \right)^{\frac{1}{2}} = (A^2 T_b)^{\frac{1}{2}} = A \sqrt{T_b}$$

$$\phi_1(t) = \frac{g_1(t)}{\|g_1(t)\|} = \frac{1}{A \sqrt{T_b}} s_1(t) = \begin{cases} \frac{1}{\sqrt{T_b}}, & 0 \leq t \leq T_b \\ 0, & \text{otherwise} \end{cases} \Rightarrow s_1(t) = A \sqrt{T_b} \phi_1(t)$$
Let \( g_2(t) = s_2(t) - \langle s_2(t), \phi_1(t) \rangle \phi_1(t) \), then

\[
\langle s_2(t), \phi_1(t) \rangle = \int_0^{T_b} s_2(t) \phi_1^*(t) dt = -\frac{A}{\sqrt{T_b}} \int_0^{T_b} dt = -A \sqrt{T_b}
\]

Hence,

\[
g_2(t) = s_2(t) - \left( -\sqrt{T_b} A \phi_1(t) \right) = s_2(t) + A \sqrt{T_b} \phi_1(t) = s_2(t) + s_1(t) = 0
\]

Therefore, \( s_2(t) = -s_1(t) = -A \sqrt{T_b} \phi_1(t) \).

This is called antipodal signaling (one binary symbol is represented by a signal which is the negative of the other).

Furthermore, only one basis function is needed to represent the two binary signals.

Consider the following correlator receiver:

\[
\sum \rightarrow x(t) \rightarrow \int_0^T (\cdot) dt \rightarrow X
\]

\[
s_i(t) \rightarrow W(t) \rightarrow \phi_i^*(t)
\]
Then the signal constellation diagram and the observation space are shown in the figure below.

![Signal Constellation Diagram](image)

Clearly, \( R_1 = \{ x : x \geq 0 \} \) and \( R_2 = \{ x : x < 0 \} \)

If the two symbols are equally likely to be transmitted, then the average probability of symbol error is given by 
\[
P_e = 1 - \frac{1}{2} \sum_{i=1}^{2} P \{ X \text{ lies in } R_i \mid m_i \text{ sent} \}.
\]
The pdf of the observation $X$, given that $m_1$ was transmitted is

$$f_{X|m_1}(x|m_1) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\frac{N_0}{2}}} e^{-\frac{(x-d_1)^2}{2(N_0/2)}} = \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(x-d_1)^2}{N_0}}, \quad d_1 = A\sqrt{T_b}$$

Then the probability of a correct decision, given that $m_1$ was transmitted is

$$P(X \in R_1|m_1) = \int_0^{\infty} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(x-d_1)^2}{N_0}} \, dx$$

Let $u = \sqrt{\frac{2}{N_0}}(x-d_1)$, $du = \sqrt{\frac{2}{N_0}} \, dx$, $\Rightarrow$ $dx = \sqrt{\frac{N_0}{2}} \, du$,

and $u^2 = \frac{(x-d_1)^2}{\frac{N_0}{2}} = 2\frac{(x-d_1)^2}{N_0} \Rightarrow \frac{u^2}{2} = \frac{(x-d_1)^2}{N_0}$. 
Hence,

\[
P(X \in R_1|m_1) = \int_{-\frac{2}{\sqrt{N_0}}}^{\frac{2}{\sqrt{N_0}}} e^{-\frac{u^2}{2}} \, du = \int_{-\frac{2}{\sqrt{2\pi}}}^{\frac{2}{\sqrt{2\pi}}} e^{-\frac{u^2}{2}} \, du = 1 - \int_{-\infty}^{\frac{2}{\sqrt{2\pi}}} e^{-\frac{u^2}{2}} \, du
\]

\[
= 1 - \int_{-\frac{2}{\sqrt{N_0}d_1}}^{\infty} e^{-\frac{u^2}{2}} \, du = 1 - Q\left(\frac{2}{\sqrt{N_0}d_1}\right) = P(X \in R_2|m_2)
\]

Therefore, the average probability of a binary error (BER) is given by

\[
P_e = 1 - \frac{1}{2} \left( \frac{2}{1} - Q\left(\frac{2}{\sqrt{N_0}d_1}\right) \right) = 1 - Q\left(\sqrt{\frac{2}{N_0}d_1}\right) = Q\left(\sqrt{\frac{2}{N_0}d_1}\right)
\]

Finally, \( d_1 = \frac{d_{12}}{2} \Rightarrow P_e = Q\left(\sqrt{\frac{2}{N_0} \frac{d_{12}}{2}}\right) = Q\left(\frac{d_{12}}{\sqrt{2N_0}}\right),
\]

where \( d_{12} \) is the distance between signal points 1 and 2.
Effect of Rotation:

Let $s_{i,r} = -s_i$ and $x_r = -s_i + W$, then

$$\|x_r - s_{i,r}\| = \|-s_i + W - (-s_i)\| = \|W\| = \|x - s_i\|, i = 1,2$$

$\Rightarrow$ the distance depends on the noise alone and $P_e$ is invariant to rotation!

Effect of Translation:

Suppose now that $s_{i,t} = s_i - a, i = 1,2$ and $x_t = x - a$, then

$$\|x_t - s_{i,t}\| = \|x - a - s_i + a\| = \|x - s_i\| = \|s_i + W - s_i\| = \|W\|$$

$\Rightarrow$ distance again depends on the noise alone and $P_e$ is invariant to translation.

Remark: Rotational invariance holds only when the rotation is caused by an orthonormal transformation matrix $Q$, i.e., for

$$Q \in \mathbb{R}^{K \times K}, \quad x_r = Qx \quad \text{and} \quad QQ^T = I_{K \times K}.$$
Let $P_e(m_i)$ be the conditional probability of symbol error when symbol $m_i$ is sent. Let $A_{ik}$, $i, k = 1,\ldots, M$, denote the event that the observation vector $x$ is closer to the signal vector $s_k$ than to $s_i$, when $m_i(s_i)$ is sent. Then

$$P_e(m_i) = P\left\{ \bigcup_{k=1 \atop k \neq i}^{M} A_{ik} \right\} \leq \sum_{k=1 \atop k \neq i}^{M} P\{A_{ik}\}, i = 1,\ldots, M.$$ 

Equality holds only when the events $A_{ik}$ are mutually exclusive. Note that $P\{A_{ik}\}$ is a pair-wise probability of a data transmission system that uses only a pair of signals, $s_i$ and $s_k$. This is different than $P\{\hat{m} = m_k | m_i\}$, the probability that the observation vector $x$ is closer to the signal vector $s_k$ than any other when $s_i(m_i)$ is sent.
Example: A message source outputs 1 of 4 symbols every $T_s$ seconds with equal probability. The 4 symbols have the signal constellation and observation space shown in the figure below.
Suppose the observation vector \( \tilde{x} \) at the input of the decision device at the receiver lies on the following region:

\[
\begin{align*}
&\{ \tilde{x} \in R' \mid m_1 \}.
\end{align*}
\]

even though \( s_1 \) was sent. Then \( P_e(m_1) = P\{ \tilde{x} \text{ lies in } R' \mid m_1 \} \).
Now, the events $A_{12}$, $A_{13}$, and $A_{14}$ are equivalent to having $\tilde{x}$ lie in each one of the following regions:
or

\[ P \{ A_{12} \} + P \{ A_{13} \} + P \{ A_{14} \} = P \{ X \text{ lies in } R'_2 \mid m_1 \} + P \{ X \text{ lies in } R'_3 \mid m_1 \} + P \{ X \text{ lies in } R'_4 \mid m_1 \} \]
\[ \geq P \{ X \text{ lies in } R' \mid m_1 \} = P_e(m_1). \]
Finally, $P\{\hat{m} = m_3 | m_1\} = P\{x \text{ lies in } R_3 | m_1\}$, where $R_3$ is depicted below
We already know that in an AWGN channel an error is caused by the noise. Moreover, WGN is identically distributed along any set of orthogonal axes. According to these criteria, an error is made when $m_i$ is sent (vector $s_i$) and $x$ lies in $R'_k$, or

$$P\{A_{ik}\} = \int_{-\infty}^{0} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(x-d_{ik}/2)^2}{N_0}} dx = \int_{-\infty}^{d_{ik}/\sqrt{2N_0}} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du = \int_{d_{ik}/\sqrt{2N_0}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du$$

where $d_{ik} \triangleq \|s_i - s_k\|$ and $u = \frac{x-d_{ik}/2}{\sqrt{N_0}}$.

Using the definition of the Q function and the error function complement, we get

$$P\{A_{ik}\} = \int_{d_{ik}/\sqrt{2N_0}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} dv = Q\left(\frac{d_{ik}}{\sqrt{2N_0}}\right) = \frac{1}{2} \text{erfc}\left(\frac{d_{ik}}{\sqrt{2N_0}}\right),$$

since $\text{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_{z}^{\infty} e^{-\lambda^2} d\lambda, z \geq 0.$
Thus,

\[ P_e(m_i) \leq \sum_{k=1}^{M} Q\left( \frac{d_{ik}}{\sqrt{2N_0}} \right), \quad i = 1, 2, \ldots, M \]

and \[ P_e = \sum_{i=1}^{M} p_i P_e(m_i) \leq \sum_{i=1}^{M} \sum_{k=1}^{M} p_i Q\left( \frac{d_{ik}}{\sqrt{2N_0}} \right), \quad p_i = P\{m_i \text{ sent}\}. \]

Up to this point, our space signal analysis has been carried out assuming a correlator receiver architecture, even though we already claimed that the optimum receiver for an AWGM channel uses a matched filter.
Consider the following matched filter detector for M-ary transmission over an AWGN channel.

Matched filter bank for M-ary signaling

Matched filter bank for M-ary signaling
Now, \( y_i(t) = \int_{-\infty}^{\infty} x(\tau)\phi_i^*(T-(t-\tau))d\tau \)

At \( t = T \), \( y_i(T) = \int_{-\infty}^{\infty} x(\tau)\phi_i^*(\tau)d\tau \).

Moreover, \( \phi_i(t) = 0 \) for \( t \notin [0,T] \). Hence,

\[
y_i(T) = X_i = \int_{0}^{T} x(\tau)\phi_i^*(\tau)d\tau,
\]

which is the same as the output of the \( i \)th branch of the correlator receiver! \( \Rightarrow \) the two receivers are equivalent. This means that we have the choice of using either a generalized correlator receiver or a bank of matched filters matched to each basis function of the set that describes the signal space. In our work, we shall mostly use correlator receiver implementations to analyze and assess communication system performance.