PASSBAND DIGITAL MODULATION TECHNIQUES

Consider the following passband digital communication system model.

\[ \cos(\omega_c t + \phi) \]

Passband digital communication system
Assumptions about the channel:

1. The channel is linear with bandwidth wide enough to accommodate the transmission bandwidth of $s_i(t)$ to minimize signal distortion at the receiver.

2. The channel noise is of the AWGN type with p.s.d. $S_w(f) = \frac{N_0}{2}, \forall f$.

Although there is a wide variety of modulation schemes, we shall mainly be concerned with Amplitude-Shift Keying (ASK), Phase-Shift Keying (PSK), Frequency-Shift Keying (FSK), and their variants. In the following figure, binary OOK is a form of ASK where the absence of signal means the transmission of a logical “0”.
Passband digital modulation examples
At the receiver end, we shall consider both coherent and noncoherent detectors. The former type needs to know both carrier frequency and phase information of the modulated waveform \( s_i(t) \), in order to demodulate the message signal. The latter type does not need to know the exact carrier frequency and phase values to demodulate the signal. The penalty paid by the noncoherent detector is that its performance is inferior than that of the coherent detector.

**Bandwidth Efficiency**

Consider \( M \)-ary signaling, with \( M = 2^n \). If we transmit blocks of \( n \) bits, then the symbol duration is \( T_s = nT_b \) and the required bandwidth is proportional to \( 1/(nT_b) \). This implies that transmission bandwidth can be reduced by the factor \( n \).

**Def.** Bandwidth efficiency is the ratio of the data rate in bits/s to the effectively utilized bandwidth. Let \( R_b \) be the bit rate and \( B \) be the effective channel bandwidth, then \( \rho = \frac{R_b}{B} \) [bits/s/Hz]

**Example:** \( \rho \) for BPSK, QPSK, 16-QAM, FSK, MSK is 1, 2, 4, < 1, 1.35, respectively.
PULSE AMPLITUDE MODULATION (PAM)

AWGN Channel:

Consider the following PAM communication system with coherent demodulation

\[
\{b_m\} \rightarrow \text{PAM Map} \rightarrow s_{pb}(t) \rightarrow \Sigma \rightarrow \times \rightarrow \{r_m\} \rightarrow \text{Decision} \rightarrow \{\hat{s}_i\}
\]

\[
\sqrt{\frac{2}{T_s}} \cos(\omega_t + \theta) \quad W(t) \quad \sqrt{\frac{2}{T_s}} \cos(\omega_t + \theta)
\]

where \(\{b_m\}\) is a stream of binary symbols, \(s_{pb}(t)\) is the band pass PAM transmitted signal, \(W(t)\) is WGN with power spectral density \(S_w(f) = \frac{N_0}{2}, \forall f\), \(\{r_m\}\) is a sequence of observed samples, and \(\{\hat{s}_i\}\) is a sequence of estimated symbols.
Let $s_i, i, \ldots, M$, be a PAM symbol that can take on $M$ possible values, with $M$ an arbitrary positive integer. Then the modulated waveform is described by

$$s_{pb}(t) = s_i \sqrt{\frac{2}{T_s}} \cos(\omega_c t + \theta) = s_i \phi_1(t), \quad \phi_1(t) = \sqrt{\frac{2}{T_s}} \cos(\omega_c t + \theta),$$

where $s_i$ is the baseband signal (symbol) that takes on the values

$$s_i = \frac{(2i - 1 - M)d}{2}, \quad i = 1, \ldots, M$$

with $d$ the intra-symbol distance.

Let $M$ be an even positive integer. Consider the following signal constellation for the PAM modulator

![Signal Constellation Diagram]
After down conversion, the detector is described by an $M$-level midrise quantizer. The decision variable at the output of the detector is described by

$$r_m = s_i + W,$$

where $W \sim G\left(0, \frac{N_0}{2}\right)$ implies $f_W(w) = \frac{1}{\sqrt{\pi N_0}} e^{-\frac{w^2}{N_0}}$ and $f_{R_m|s_i}(r_m | s_i) = \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(r_m-s_i)^2}{N_0}}$.

The quantizer receiver is equivalent to the minimum distance receiver when the ML decision criterion is applied. Therefore, for the inner points $(i = 2, \ldots, M-1)$, the probability of a correct decision (cd) given that $s_i$ was transmitted is given by

$$P\{cd | s_i\} = \int_{R_i} f_{R_m|s_i}(r_m | s_i) dr_m = \int_{s_i-d/2}^{s_i+d/2} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(r_m-s_i)^2}{N_0}} dr_m$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\frac{d^2}{2N_0}}^{\frac{d^2}{2N_0}} e^{-\frac{u^2}{2}} du = 1 - \left[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{d^2}{2N_0}} e^{-\frac{u^2}{2}} du \right] + \int_{\frac{d^2}{2N_0}}^{\infty} e^{-\frac{u^2}{2}} du$$

$$= 1 - 2Q\left(\sqrt{\frac{d^2}{2N_0}}\right), \quad i = 2, \ldots, M-1.$$
Where it has been assumed that all symbols occur with equal probability, i.e.,

\[ P\{S_i\} = \frac{1}{M}, \quad i = 1, \ldots, M \]  

(the decision regions boundaries are equidistant for the \(M-2\) innermost constellation points).

For \(i = 1\):

\[
P\{cd|s_1\} = \int_{R_1} f_{r_{m|s_1}}(r_m | s_1) dr_m = \int_0^{s_1 + \frac{d}{2}} \frac{1}{\sqrt{\pi N_0}} e^{-(r_m - s_1)^2 / N_0} dr_m = \int_{-\infty}^{\sqrt{d^2 / 2N_0}} e^{-u^2/2} du = 1 - Q\left(\frac{\sqrt{d^2}}{\sqrt{2N_0}}\right).
\]

Likewise, for \(i = M\),

\[
P\{cd|s_M\} = P\{cd|s_1\} = 1 - Q\left(\frac{\sqrt{d^2}}{\sqrt{2N_0}}\right).
\]

Let \(P_{cd} \triangleq P\{cd\}\), then

\[
P_{cd} = \sum_{i=1}^{M} P\{cd, S_i\} = \sum_{i=1}^{M} P\{cd|S_i\} P\{S_i\} = \frac{1}{M} \sum_{i=1}^{M} P\{cd|S_i\}.
\]
The probability of a correct decision is then given by

\[
P_{cd} = 2 \sum_{i=1}^{M} P\{cd | S_i\} P\{S_i\}
\]

\[
= 2P\{cd | S_1\} P\{S_1\} + 2 \sum_{i=2}^{M} P\{cd | S_i\} P\{S_i\}
\]

\[
= 2 \left[ 1 - Q\left( \frac{d^2}{2N_0} \right) \right] P\{S_1\} + 2 \sum_{i=2}^{M} 1 - 2Q\left( \frac{d^2}{2N_0} \right) P\{S_i\}.
\]

Rewriting the last equation, we get

\[
P_{cd} = 2P\{S_1\} + 2 \sum_{i=2}^{M} P\{S_i\} - 2 \left[ P\{S_1\} + 2 \sum_{i=2}^{M} P\{S_i\} \right] Q\left( \frac{d^2}{2N_0} \right).
\]
\[ P_{cd} = 1 - 2 \left[ 2P\{S_1\} + 2 \sum_{i=2}^{M} P\{S_i\} - P\{S_1\} \right] Q\left( \sqrt{\frac{d^2}{2N_0}} \right) = 1 - 2 \left[ 1 - \frac{1}{M} \right] Q\left( \sqrt{\frac{d^2}{2N_0}} \right). \]

The symbol error probability is therefore

\[ P_{se} = 1 - P_{cd} = 2 \left[ 1 - \frac{1}{M} \right] Q\left( \sqrt{\frac{d^2}{2N_0}} \right) = 2 \left( \frac{M - 1}{M} \right) Q\left( \sqrt{\frac{d^2}{2N_0}} \right), \]

Now, let’s compute the average symbol energy \( \mathcal{E}_s \).

\[ \mathcal{E}_s = E\{S^2\} = \sum_{i=1}^{M} S_i^2 P\{S_i\} = 2 \sum_{i=1}^{M} S_i^2 P\{S_i\} = 2 \sum_{i=1}^{M} \frac{(2i-1-M)^2}{4} d^2 P\{S_i\} = \frac{d^2}{2M} \sum_{i=1}^{M} (2i-1-M)^2 \]

Let \( K \triangleq \frac{1}{M} \sum_{i=1}^{M} (2i-1-M)^2 \), then \( K = (M+1) \left[ \frac{1}{2} (M+1) - \frac{2}{3} \left( \frac{M}{2} + 1 \right) \right] \), \( \frac{d^2}{2} = \frac{\mathcal{E}_s}{K} \) and the symbol error probability can be rewritten as

\[ P_{se} = 2 \left[ \frac{M - 1}{M} \right] Q\left( \sqrt{\frac{\mathcal{E}_s}{KN_0}} \right). \]
For $M = 16$, $K = 42.5$ and $P_{se} = \left[ \frac{15}{8} \right] Q \left( \sqrt{\frac{4E_b}{42.5N_0}} \right) = 1.875Q \left( \sqrt{\frac{0.094E_b}{N_0}} \right)$. 

![Graph showing symbol error probability vs. $E_b/N_0$ in dB]
**QUADRATURE AMPLITUDE MODULATION (QAM)**

**AWGN Channel:**

In quadrature amplitude modulation both amplitude and phase are modulated. Hence, the resulting waveform will have both an in-phase and a quadrature component. As before, increasing the distance between constellation points will improve system performance at the expense of higher average transmitted power.

Let the passband transmitted signal be described by

\[ \tilde{s}_{pb}(t) = \Re \{ (a_i + jb_i) e^{j(\omega_c t + \theta_c)} \} = a_i \cos(\omega_c t + \theta_c) - b_i \sin(\omega_c t + \theta_c), \quad nT_s \leq t \leq (n+1)T_s, \]

where \( a_i \) and \( b_i \) are the in-phase and quadrature components, respectively, of the \( i^{th} \) symbol, \( i = 1,\ldots,M \). An 8-QAM signal constellation is shown in the next figure.
8-QAM signal constellation
A sample 8-QAM waveform is shown next. It is clear from the figure that the QAM modulated waveform does not have constant envelope. Hence, the transmitter needs to use a highly linear power amplifier and the receiver needs an automatic gain control circuit to compensate for the amplitude variations due to channel imperfections.

Unmodulated carrier and 8-QAM modulated signals
A 64-QAM signal constellation is shown below.

64-QAM signal constellation
Let the complex equivalent baseband transmitted signal be described by

\[ s_{bb}(t) = a_i \phi_1(t) + b_i \phi_2(t), \quad nT_s \leq t \leq (n+1)T_s. \]

where \( \phi_1(t) = \sqrt{\frac{2}{T_s}} \cos(\omega_c t + \theta) \) and \( \phi_2(t) = \sqrt{\frac{2}{T_s}} \sin(\omega_c t + \theta) \).

Then, the complex baseband equivalent signal (two-dimensional interpretation) seen at the input of the receiver decision device (after bandpass filtering and demodulation by a correlator receiver) is described by

\[
\tilde{r}(t) = s_i + \tilde{n}(t) = a_i + jb_i + \tilde{n}(t)
\]
\[
= a_i + n_I(t) + j\left[b_i + n_Q(t)\right], \quad nT_s \leq t \leq (n+1)T
\]

Performance will depend on the type (shape) of constellation that is used at the transmitter and on the detection scheme.

Regardless of the constellation shape, if the detector selects symbol \( s_j \) when the observed measurement \( \tilde{r}(t) \) lies on region \( \mathcal{R}_j \) of the observation space (equivalent to minimum Euclidean distance detection), then the system performance can be obtained as follows:
Suppose that at the $n$th signaling period signal $s_i(t)$ is transmitted. Then the probability of a correct decision, given that symbol $s_i$ was transmitted, is described by

$$P\{cd \mid S_i\} = \iint_{\mathbb{R}_i} f\left(r_1, r_2 \mid S_i\right) dr_1 dr_2,$$

where

$$r_1 \triangleq \Re \{\tilde{r}(t)\} = a_i + n_I(nT_s)$$

$$r_2 \triangleq \Im \{\tilde{r}(t)\} = b_i + n_Q(nT_s).$$

Let $\tilde{r} \triangleq \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$ and $\tilde{n} \triangleq \begin{bmatrix} n_I \\ n_Q \end{bmatrix}$.

With $E\{n_I\} = E\{n_Q\} = 0, E\{n_I^2\} = E\{n_Q^2\} = N_0$ and $E\{n_In_Q\} = 0$.

Define $P\{cd \mid S_i\}_j \equiv P\left\{\|\tilde{r} - S_i\|^2 \leq \|\tilde{r} - S_j\|^2 \mid S_i\right\}$, $j \neq i; \quad i, j = 1, \ldots, M$ as the probability that the observed vector $\tilde{r}$ is closer to constellation point $\tilde{S}_i$ than to $\tilde{S}_j$, given that symbol $\tilde{S}_i$ was transmitted.
Conversely, define the probability of an incorrect decision (id), given that symbol $S_i$ was transmitted as

$$P\{id \mid S_i\}_{j} = P\left\{\|\tilde{\mathbf{r}} - S_i\|^2 > \|\tilde{\mathbf{r}} - S_j\|^2 \mid S_i\} \right\}, j \neq i; i, j = 1, \ldots, M$$

But, when symbol $S_i$ is transmitted, $\|\tilde{\mathbf{r}} - S_i\|^2 = \|\tilde{\mathbf{n}}\|^2 = n_i^2 + n_Q^2$ and

$$\|\tilde{\mathbf{r}} - S_j\|^2 = (a_i - a_j)^2 + (b_i - b_j)^2 + 2(a_i - a_j)n_i + 2(b_i - b_j)n_Q + n_i^2 + n_Q^2$$

Hence, $\left\{\|\tilde{\mathbf{r}} - S_i\|^2 > \|\tilde{\mathbf{r}} - S_j\|^2 \mid S_i\right\}$ implies

$$n_i^2 + n_Q^2 > (a_i - a_j)^2 + (b_i - b_j)^2 + 2\left[(a_i - a_j)n_i + (b_i - b_j)n_Q\right] + n_i^2 + n_Q^2$$

or

$$(a_j - a_i)n_i + (b_j - b_i)n_Q > \frac{1}{2}\left[(a_i - a_j)^2 + (b_i - b_j)^2\right].$$

The left hand side of the last equation is a linear combination of zero-mean Gaussian random variables.
Let $v_{ij} \triangleq (a_j - a_i)n_I + (b_j - b_i)n_Q$ and $C_{ij} \triangleq \frac{1}{2}\left[ (a_i - a_j)^2 + (b_i - b_j)^2 \right]$ then $v_{ij} \sim G\left(0, \sigma_{v_{ij}}^2\right)$ where the variance $\sigma_{v_{ij}}^2$ is given by

$$\sigma_{v_{ij}}^2 = \left[ (a_i - a_j)^2 + (b_i - b_j)^2 \right] N_0$$

$$= 2C_{ij} N_0.$$ 

Hence,

$$P\{id \mid S_i\}_j = P\{v_{ij} > C_{ij}\} = \frac{1}{\sqrt{2\pi \sigma_{v_{ij}}}} \int_{C_{ij}}^{\infty} e^{-\frac{v_{ij}^2}{2\sigma_{v_{ij}}^2}} dv_{ij}$$

$$= Q\left(\frac{C_{ij}}{\sigma_{v_{ij}}}\right) = Q\left(\sqrt{\frac{C_{ij}}{2N_0}}\right) = Q\left(\frac{\sqrt{(a_i - a_j)^2 + (b_i - b_j)^2}}{4N_0}\right)$$

$$\triangleq P\{error \mid S_i\}_j, \ j \neq i; \ i, j = 1, \ldots, M$$

and

$$P\{error \mid S_i\} \leq \sum_{\substack{j=1 \ j \neq i}}^{M} P\{error \mid S_i\}_j, i = 1, \ldots, M.$$
Finally, an upper bound of the average symbol error rate for QAM is given by

\[
P_{se} = \sum_{i=1}^{M} P\{\text{error} \mid \mathcal{S}_i \} P\{\mathcal{S}_i \} \leq \sum_{i=1}^{M} \sum_{j=1}^{M} \sum_{j \neq i} Q \left( \frac{C_{ij}}{\sigma_{v_{ij}}} \right) P\{\mathcal{S}_i \}
\]

Note that this last equation is an upper bound of the symbol error probability when the decision metric is the minimum Euclidean distance and the probability of symbol transmission is arbitrary. Also, \( M \) can be an arbitrary odd or even positive integer.
Consider now a rectangular 16-QAM signal constellation. Let the distance to the nearest neighbor be \( d = 2a \). Then the constellation diagram is shown in the next figure.
Suppose all 16 symbols are transmitted with equal probability, i.e.

\[ P\{m_i\} = \frac{1}{16}, i = 1, 2, \ldots, 16, \]

then the average transmitted energy per symbol can be computed as follows:

\[ E_{s,av} = \sum_{i=1}^{16} E_i \cdot P\{m_i\}, \]

where \( E_i = d_i^2 \) is the energy associated with the \( i^{th} \) symbol and \( d_i \) is the distance from the origin to the \( i^{th} \) point on the constellation. From the figure we can see that the 4 innermost points have the same distance \( d_1 \), the outer 8 rectangle points along the \( \phi_1 \) and \( \phi_2 \) axes have distance \( d_2 \), and the outer 4 corner points have distance \( d_3 \), where

\[ d_1^2 = a^2 + a^2 = 2a^2, \]
\[ d_2^2 = (3a)^2 + a^2 = 10a^2, \]
\[ d_3^2 = (3a)^2 + (3a)^2 = 18a^2. \]

The average transmitted energy per symbol is therefore

\[ E_{s,av} = 4\left[ \frac{1}{16} (2a^2) \right] + 8\left[ \frac{1}{16} (10a^2) \right] + 4\left[ \frac{1}{16} (18a^2) \right] \]
\[ = \frac{1}{16} \left[ 8a^2 + 80a^2 + 72a^2 \right] = 10a^2. \]
The nearest neighbor intra-symbol distance is \( d = 2a \). Hence, \( E_{s,av} = 2.5d^2 \) and \( d = \sqrt{\frac{2}{5}} E_{s,av} \).

System performance in AWGN can now be assessed. Assuming equal probability of symbol transmission, we can use the maximum likelihood detection approach (minimization of distance). In this case, the decision region for one of the inner points (symbols) is shown in the next figure.

Decision region for symbol \( m_i \) when all symbols occur with equal probability.
The probability of a correct decision, given that this symbol was transmitted is described by

$$P\{\text{correct decision} | \text{given } m_i\} = \int \int_{R_i} f_x(x|m_i) \, dx_1 \, dx_2$$

Let $P_c$ be the average probability of a correct decision. Then

$$P_c = \sum_{i=1}^{16} P\{\text{choose } m_i \cap m_i \text{ sent}\} = \sum_{i=1}^{16} P\{\text{choose } m_i \mid m_i \text{ sent}\} P\{m_i \text{ sent}\}.$$

If $P\{m_i \text{ sent}\} = \frac{1}{16}$, $\forall i$, then

$$P_c = \frac{1}{16} \sum_{i=1}^{16} P\{\text{choose } m_i \mid m_i \text{ sent}\} = \frac{1}{16} \sum_{i=1}^{16} P\{X \in R_i \mid m_i \text{ sent}\}.$$

But, $P\{X \in R_i \mid m_i \text{ sent}\} = 1 - P\{X \notin R_i \mid m_i \text{ sent}\}$. Therefore,

$$P_c = \frac{1}{16} \sum_{i=1}^{16} (1 - P\{X \notin R_i \mid m_i \text{ sent}\}) = 1 - \frac{1}{16} \sum_{i=1}^{16} P\{X \notin R_i \mid m_i \text{ sent}\}.$$
Let $P_e$ be the average probability of symbol error (SER), then

$$SER = P_e = P\{\text{symbol error}\} = 1 - P_e = \frac{1}{16} \sum_{i=1}^{16} P\{X \notin R_i \mid m_i \text{ sent}\}.$$ 

For a sufficiently high SNR, an error will occur if $m_i$ is confused with one of its nearest neighbors.

Let $R^i_j$ be the decision region of one of the nearest neighbors of symbol $m_i$. Then, for the 4 innermost points of the constellation (symbols) $j = 1, 2, 3, 4$; for the 8 edge points (excluding the corner points) $j = 1, 2, 3$; and for the 4 corner points $j = 1, 2$. 
Consider now the 4 innermost points. Then using the union bound, we get

$$P\{X \not\in R_i \mid m_i \text{ sent}\} \leq P\left\{X \in \bigcup_{j=1}^{4} R_j^i \mid m_i \text{ sent}\right\} \leq \sum_{j=1}^{4} P\{X \in R_j^i \mid m_i \text{ sent}\}.$$

For the 8 edge points,

$$P\{X \not\in R_i \mid m_i \text{ sent}\} \leq P\left\{X \in \bigcup_{j=1}^{3} R_j^i \mid m_i \text{ sent}\right\} \leq \sum_{j=1}^{3} P\{X \in R_j^i \mid m_i \text{ sent}\}.$$

For the 4 corner points,

$$P\{X \not\in R_i \mid m_i \text{ sent}\} \leq P\left\{X \in \bigcup_{j=1}^{2} R_j^i \mid m_i \text{ sent}\right\} \leq \sum_{j=1}^{2} P\{X \in R_j^i \mid m_i \text{ sent}\}.$$

In conclusion, for any point on the 16-QAM constellation, a loose upper bound for the symbol error rate (SER) for high SNR is given by

$$P\{X \not\in R_i \mid m_i \text{ sent}\} \leq \sum_{j=1}^{4} P\{X \in R_j^i \mid m_i \text{ sent}\}.$$
Assuming Grey encoding of the QAM symbols, \( P\{X \in R^i_j \mid m\text{ sent}\} \) is the same as the pair-wise bit error probability, i.e.

\[
P\{X \in R^i_j \mid m\text{ sent}\} = P\left\{X_i > \frac{d}{2}\right\} = P\left\{X_i < -\frac{d}{2}\right\}, \quad i = 1, 2
\]

\[
= Q\left(\sqrt{\frac{d^2}{2N_0}}\right) = Q\left(\sqrt{\frac{2E_{s,av}}{5N_0}}\right) = Q\left(\sqrt{\frac{E_{s,av}}{5N_0}}\right) = Q\left(\sqrt{\frac{4E_{b,av}}{5N_0}}\right).
\]

and

\[
P\{X \not\in R_i \mid m\text{ sent}\} \leq \sum_{j=1}^{4} Q\left(\sqrt{\frac{4E_{b,av}}{5N_0}}\right) = 4Q\left(\sqrt{\frac{4E_{b,av}}{5N_0}}\right).
\]

Finally, the average symbol error probability (SER) is upper bounded by

\[
SER = P_e \leq \frac{1}{16} \sum_{j=1}^{16} 4Q\left(\sqrt{\frac{4E_{b,av}}{5N_0}}\right) = 4Q\left(\sqrt{\frac{4E_{b,av}}{5N_0}}\right).
\]
In general, for $M = 2^k$ and a rectangular signal constellation with symbols transmitted with equal probability and Grey encoding,

$$SER = P_e \leq 4Q\left(\sqrt{\frac{d^2}{2N_0}}\right),$$

where $d$ is the nearest neighbor intra-symbol distance, which can be expressed in terms of the average symbol energy $E_s$, which must be computed for every specific signal constellation. Other tighter upper bounds can be found in the literature.

The following figure shows the SER performance of a 16 QAM digital communication system that uses a rectangular constellation that has the same intra-symbol distance $d$. 
16-QAM SER performance in AWGN