Response Gradients for Nonlinear Beam-Column Elements under Large Displacements

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Abstract: Accurate and efficient response gradient computations for nonlinear geometry are required in structural reliability, optimization, system identification, and response sensitivity analysis of frame structures that undergo large displacements. In this paper, the exact response gradient of beam-column finite elements under large displacements is derived considering uncertain material and geometric parameters. The element response formulation takes place in a corotational reference frame that displaces and rotates with the element, thus permitting the separation of nonlinear material from nonlinear geometric effects in the computation of the response, as well as of the gradient of the response. Relative to the corotational reference frame, small deformation theory suffices for all structural engineering applications. Thus, the proposed response gradient computations are applicable to most beam-column element formulations available in the literature, including force-based and mixed formulations.

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Introduction

A comprehensive performance-based design methodology necessitates the computation of structural response until collapse. Under these conditions, large displacement beam-column theories are required for an accurate assessment of the response of frame structures. Frame structures are prone to collapse when column members carrying high axial loads experience significant drift, or when vertical load carrying members are removed from the structure due to accident or explosion. The corotational theory (Crisfield 1991) is an ideal approach for describing the response of frames under large displacements because it permits the separation of the nonlinear geometric response of a beam-column element from its nonlinear material response. In this approach, the material response is described in a reference frame that displaces and rotates with the element. Relative to the corotational frame, the assumption of small deformations is valid in computing the material response because it is always possible to subdivide the structural member into smaller elements. Thus, beam-column finite element models for small deformations can be used to describe the nonlinear material response, in particular distributed inelasticity elements based on a displacement, force, or mixed formulation as summarized, e.g., by Alemdar and White (2005).

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Algorithms in structural reliability, optimization, and system identification require that the gradient of the structural response be computed for a set of uncertain parameters in order to converge to the optimal solution point (Liu and Der Kiureghian 1991). Without the derivative of the response of beam-column elements that undergo large displacements of the corotational reference frame, it is not possible to deploy such elements in gradient-based applications or in stand-alone response sensitivity analysis. In such applications the direct differentiation method (Kleiber et al. 1997) is computationally superior to the finite difference method because it permits the exact response gradient to be computed as the simulation proceeds rather than by approximation through repeated simulations. Moreover, the latter is significantly more time consuming because it requires that the entire simulation be repeated for each uncertain parameter in the model, and it is prone to round-off error for small perturbations of a parameter value.

The objective of this paper is to use the direct differentiation method (DDM) in deriving the exact response gradient for beam-column elements under large displacements of the corotational reference frame. The derivation applies to all beam-column formulations for material nonlinear behavior within the corotating frame; however, particular attention is paid to the force-based formulation. The proposed formulation of the response gradient is based on the following assumptions:

- The displacements of the corotational reference frame (and thus the translations and rotations of the nodes) are assumed to be large, requiring the element equilibrium equations to be written in the deformed configuration of the element.
- The corotational formulation is limited to the two-dimensional case to avoid the complex transformation of large rotations in three dimensions. The derivation of the response gradient can be extended to the three-dimensional case, but the expressions are significantly more involved and do not add to the underlying concepts.
- The element deformations in the corotational reference frame are assumed to be small. Thus, small strain theory describes

the material nonlinear response at a cross section of the beam-column element.

- Dynamic equilibrium effects are taken into account at the global level of the structure rather than at the local element level. As a result, only the static resisting forces of the element are considered in the response gradient equations derived herein.

This paper begins with the global formulation for computing the gradient of the structural response. Then the corotational theory is described, followed by the exact differentiation of the governing kinematic and equilibrium equations. Next, the response gradient of force-based beam-column elements is derived considering both material and geometric uncertainties within the corotating frame of reference. The paper concludes with numerical examples that validate the exact response gradient for the corotational theory in conjunction with the force-based element formulation of distributed inelasticity.

Equations for the Global Response Gradient

The analysis of a structural system by the finite element method depends on assumptions regarding the properties of the system. These properties, defined in terms of uncertain parameters, describe the loads applied to the structure, the materials that make up the structural members, and the location of supports, connections, and joints. The load, material, and geometric parameters for the structure are collected in a vector, $\mathbf{\Theta}$, and the global equations of static equilibrium take the form

$$\mathbf{P}_f(\mathbf{\Theta},t) - \mathbf{P}_r(\mathbf{U}(\mathbf{\Theta},t),\mathbf{\Theta}) = \mathbf{0} \tag{1}$$

Eq. (1) indicates that equilibrium is attained when the internal resisting forces in the vector $\mathbf{P}_r$ balance the applied external loads in the vector $\mathbf{P}_f$. This external load vector varies with pseudotime, t, and is an explicit function of the parameters in $\mathbf{\Theta}$ that describe the loads applied to the structure. The vector $\mathbf{P}_r$ represents the static resisting forces of the structure, and it is assembled from element contributions by standard finite element procedures (Zienkiewicz and Taylor 2000). The resisting force vector is a nonlinear function of the nodal displacement vector, $\mathbf{U}$, which is a function of all the parameters in $\mathbf{\Theta}$ as a change in any load, material, or geometric property will affect the structural response. The implicit solution for $\mathbf{U}$ in Eq. (1) at each time step during the structural simulation is achieved by a nonlinear root-finding algorithm, such as the Newton-Raphson method (Stoer and Bulirsch 1993). In addition to its implicit dependence on $\mathbf{\Theta}$ through the nodal displacement vector, the resisting force vector is an explicit function of the parameters in $\mathbf{\Theta}$ that describe the material and geometric properties of the structural system. Inertial and damping forces are omitted from Eq. (1) because the inclusion of dynamic equilibrium effects, as well as the associated computation of the response gradient, is straightforward for the common approaches of lumped mass and classical damping (Kleiber et al. 1997; Franchin 2004).

To formulate the governing equations for the gradient of the structural response, Eq. (1) is differentiated by the chain rule with respect to a particular parameter, θ, from the entire collection of parameters in the vector $\mathbf{\Theta}$

$$\frac{\partial \mathbf{P}_f}{\partial \theta} - \left(\frac{\partial \mathbf{P}_r}{\partial \mathbf{U}} \frac{\partial \mathbf{U}}{\partial \theta} + \left.\frac{\partial \mathbf{P}_r}{\partial \theta}\right|_{\mathbf{U}} \right) = \mathbf{0} \tag{2}$$

The vector $\partial \mathbf{P}_r/\partial\theta|_{\mathbf{U}}$=conditional derivative of the resisting force vector, and it is assembled from element contributions in the same manner as the resisting force vector itself. Physically, this vector represents the forces that must be applied to the structure to keep the nodal displacements fixed due to changes in the parameter θ. The vector $\partial \mathbf{P}_f/\partial\theta$ is the gradient of the applied load vector and it is nonzero only when θ refers to an uncertain load parameter. To solve for the gradient of the nodal response, $\partial \mathbf{U}/\partial\theta$, Eq. (2) is rewritten as the following system of linear equations

$$\mathbf{K}_T \frac{\partial \mathbf{U}}{\partial \theta} = \frac{\partial \mathbf{P}_f}{\partial \theta} - \left.\frac{\partial \mathbf{P}_r}{\partial \theta}\right|_{\mathbf{U}} \tag{3}$$

where $\mathbf{K}_T=\partial \mathbf{P}_r/\partial \mathbf{U}$=tangent stiffness matrix of the structure and is assembled from element contributions.

For path-dependent structural response, the response gradient is also path dependent. In addition to the history variables that track the path-dependent behavior of each element, it is necessary to store the gradient of these history variables with respect to each parameter in $\mathbf{\Theta}$. Zhang and Der Kiureghian (1993) describe a two-phase process for path-dependent response gradient computations. The assembly of the right-hand side of Eq. (3) and the solution of the linear system of equations gives $\partial \mathbf{U}/\partial\theta$ in phase one. In phase two, each element uses the gradient of the nodal response to update the gradients of its internal history variables. For computational efficiency when using direct equation solvers (Golub and Van Loan 1996; Demmel 1997), the factorization of $\mathbf{K}_T$ can be reused to determine $\partial \mathbf{U}/\partial\theta$ for each parameter in the vector $\mathbf{\Theta}$ during phase one of this two-phase process.

Corotational Geometric Theory

The corotational theory (Crisfield 1991) for two-dimensional beam-column elements is summarized in this section and follows the presentation of a basic system within the corotating frame of reference (Filippou and Fenves 2004). There are three components to the corotational theory: (1) The transformation of the element displacements and forces between the global and local coordinate systems, (2) the removal of the rigid body modes from the element displacement field, and (3) the equilibrium relationship between the forces in the basic and local coordinate systems. The identification of these three components will facilitate the derivation of the response gradients for the corotational theory.

Transformation from Global to Local Coordinates

The first component of the corotational theory is the transformation of the element displacements from the global coordinate system to displacements in a coordinate system that coincides with the local axes of the element, as shown in Fig. 1. This transformation is described by the matrix-vector product

$$\mathbf{u} = \mathbf{a}_r \mathbf{u}_e \tag{4}$$

where $\mathbf{u}_e$ and $\mathbf{u}$=element displacement vectors in the global and local coordinate systems, respectively. The matrix $\mathbf{a}_r \equiv \mathbf{a}_r(\theta)$ transforms the nodal translations and rotations between the coordinate systems and is a function of only the undeformed element configuration in the global coordinate system.

The vector of element forces in the local coordinate system, $\mathbf{p}$, transforms to the vector of element forces in the global coordinate system, $\mathbf{p}_e$, by the contragradient relationship

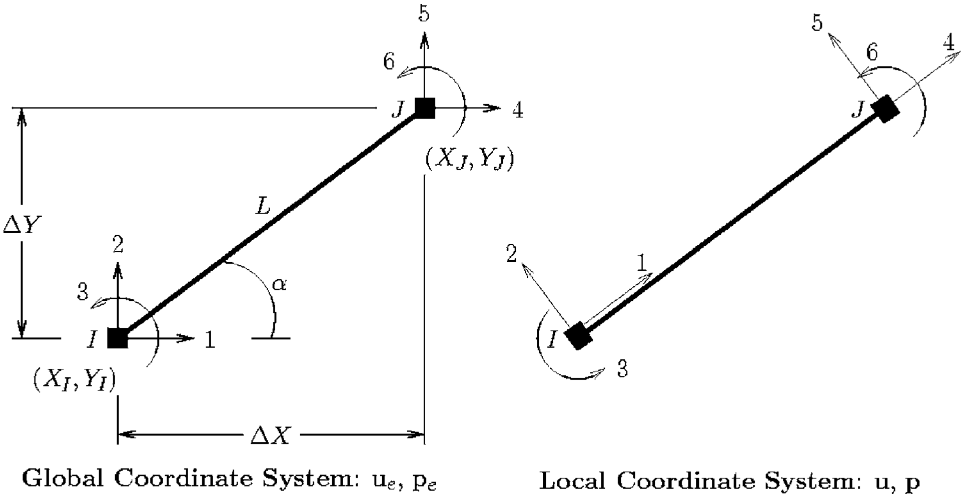

Fig. 1. Element degrees of freedom in the global and local coordinate systems

$$\mathbf{p}_e = \mathbf{a}_r^T \mathbf{p} \tag{5}$$

where, similar to $\mathbf{P}_r$, the element forces depend on any parameter in $\mathbf{\Theta}$ in one of two ways: Explicitly when θ represents a material or geometric property of the element, or implicitly through the nodal displacements, i.e., $\mathbf{p}_e \equiv \mathbf{p}_e(\mathbf{u}_e(\theta), \theta)$ and $\mathbf{p} \equiv \mathbf{p}(\mathbf{u}(\theta), \theta)$. The element displacement vectors, $\mathbf{u}_e \equiv \mathbf{u}_e(\theta)$ and $\mathbf{u} \equiv \mathbf{u}(\theta)$, depend on every material, geometric, and load parameter of the structural system because the element displacements are selected directly from the nodal displacement vector, $\mathbf{U}$, which is a function of all parameters in $\mathbf{\Theta}$.

Removal of Rigid Body Displacement Modes

The second component of the corotational theory is the removal of rigid body modes from the element displacement field. For beam-column elements in two dimensions, there are six displacement modes, three of which correspond to rigid body motion. The removal of the rigid body modes produces the element deformation vector $\mathbf{v} \equiv \mathbf{v}(\mathbf{u}(\theta), \theta)$. The explicit dependence of $\mathbf{v}$ on θ is due to the parameters that correspond to the nodal coordinates of the element, whereas the vector $\mathbf{u}$ introduces an implicit dependence on θ.

The three element deformations can be expressed in terms of the end displacements in the local coordinate system

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} L_n - L \\ u_3 - \beta \\ u_6 - \beta \end{bmatrix} \tag{6}$$

The first deformation, v_1, is the change in length of the element, where L=undeformed length, computed from the nodal coordinates in the undeformed configuration and L_n=deformed length

$$L_n = \sqrt{(L + \Delta u_x)^2 + (\Delta u_y)^2} \tag{7}$$

where $\Delta u_x = u_4 - u_1$ and $\Delta u_y = u_5 - u_2$ are the relative nodal displacements in the local coordinate system, as shown in Fig. 2.

The second and third element deformations, v_2 and v_3, represent the rotation of the tangent to the deformed shape at nodes I and J, respectively, relative to the rigid body rotation, β, of the element chord

$$\beta = \arctan \frac{\Delta u_y}{L + \Delta u_x} \tag{8}$$

With the removal of the rigid body displacement modes from the element displacement vector complete, attention turns to the equilibrium relationship between the element forces in the basic system and the forces in the local coordinate system.

Equilibrium Relationship of the Basic and Local Forces

The basic force vector, $\mathbf{q} \equiv \mathbf{q}(\mathbf{v}(\theta), \theta)$, contains the work conjugates to the element deformations and is a nonlinear function of $\mathbf{v}$ and the parameter θ. The axial force and two end moments shown in Fig. 3 are the basic forces in the simply supported system. The element state determination procedure computes the basic forces for given element deformations depending on the formulation of nonlinear material response, as described later in this paper for the force-based formulation.

The satisfaction of element equilibrium in the deformed configuration in Fig. 3 gives the relationship between the basic forces and the forces in the local coordinate system. This relationship takes the matrix-vector form

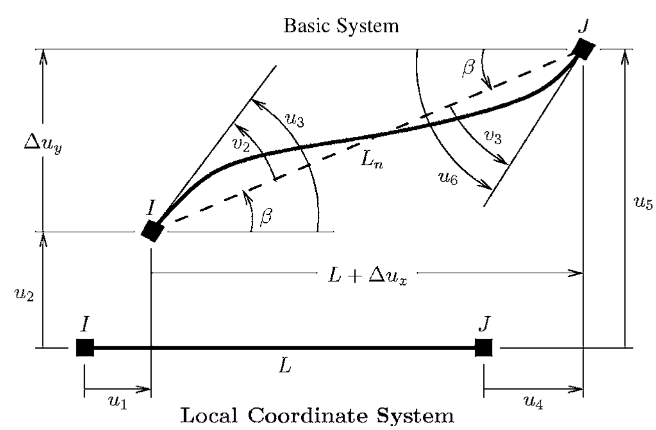

Fig. 2. Geometric transformation between the local coordinate system and the basic system

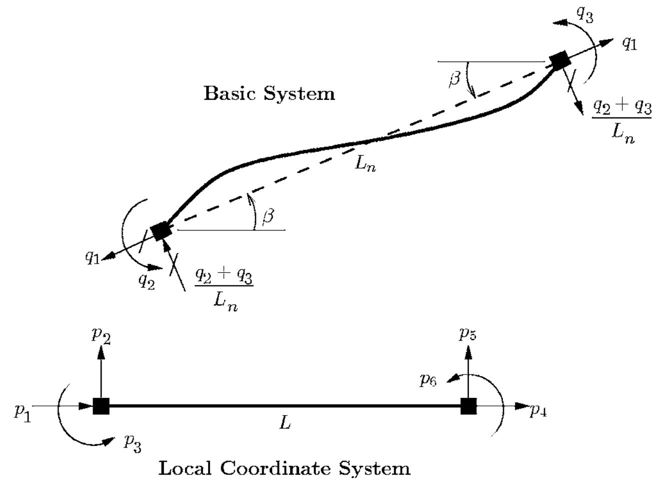

Fig. 3. Equilibrium transformation between the basic system and the local coordinate system

$$\mathbf{p} = \mathbf{a}^T\mathbf{q} \tag{9}$$

where the equilibrium transformation matrix is

$$\mathbf{a} = \begin{bmatrix} -\cos\beta & -\sin\beta & 0 & \cos\beta & \sin\beta & 0 \\ -\sin\beta/L_n & \cos\beta/L_n & 1 & \sin\beta/L_n & -\cos\beta/L_n & 0 \\ -\sin\beta/L_n & \cos\beta/L_n & 0 & \sin\beta/L_n & -\cos\beta/L_n & 1 \end{bmatrix} \tag{10}$$

with $\cos\beta = (L+\Delta u_x)/L_n$ and $\sin\beta = \Delta u_y/L_n$, as shown in Fig. 2. It can be shown that the equilibrium transformation matrix is equal to the partial derivative of the element deformations with respect to the element displacements, i.e., $\mathbf{a} = \partial\mathbf{v}/\partial\mathbf{u}$. The element stiffness matrix in the local coordinate system, $\mathbf{k}_l = \partial\mathbf{p}/\partial\mathbf{u}$, is obtained by applying the chain and product rules to Eq. (9)

$$\mathbf{k}_l = \mathbf{a}^T\mathbf{k}\mathbf{a} + \frac{\partial\mathbf{a}^T}{\partial\mathbf{u}}\mathbf{q} \tag{11}$$

where $\mathbf{k} = \partial\mathbf{q}/\partial\mathbf{v}$ = stiffness matrix in the basic system and $\partial\mathbf{a}/\partial\mathbf{u} = \partial^2\mathbf{v}/\partial\mathbf{u}^2$. The first term on the right-hand side of Eq. (11) is the material contribution to the element stiffness matrix, while the second term represents the geometric contribution. After transformation by the Givens rotations in $\mathbf{a}_r$, the element stiffness matrix in the global coordinate system, $\mathbf{k}_e = \mathbf{a}_r^T\mathbf{k}_l\mathbf{a}_r$, is assembled into $\mathbf{K}_T$ in Eq. (3).

Exact Differentiation of the Corotational Geometric Theory

With the three components of the corotational formulation defined, attention now turns to their differentiation with respect to θ in order to determine the contribution of the element resisting forces to the gradient of the structural response.

Differentiation of the Global to Local Transformation

To determine the element contribution, $\partial\mathbf{p}_e/\partial\theta|_{\mathbf{u}_e}$, to the conditional derivative of the structural resisting force vector, $\partial\mathbf{P}_r/\partial\theta|_{\mathbf{U}}$, it is necessary to differentiate with respect to θ the element equilibrium relationship defined in Eq. (5)

$$\frac{\partial\mathbf{p}_e}{\partial\theta} = \mathbf{a}_r^T\frac{\partial\mathbf{p}}{\partial\theta} + \frac{\partial\mathbf{a}_r^T}{\partial\theta}\mathbf{p} \tag{12}$$

Then, by applying the chain rule in the differentiation of $\mathbf{p}_e \equiv \mathbf{p}_e(\mathbf{u}_e(\theta),\theta)$ and $\mathbf{p} \equiv \mathbf{p}(\mathbf{u}(\theta),\theta)$ with respect to θ, Eq. (12) expands to

$$\mathbf{k}_e\frac{\partial\mathbf{u}_e}{\partial\theta} + \left.\frac{\partial\mathbf{p}_e}{\partial\theta}\right|_{\mathbf{u}_e} = \mathbf{a}_r^T\left(\mathbf{k}_l\frac{\partial\mathbf{u}}{\partial\theta} + \left.\frac{\partial\mathbf{p}}{\partial\theta}\right|_{\mathbf{u}}\right) + \frac{\partial\mathbf{a}^T}{\partial\theta}\mathbf{p} \tag{13}$$

where $\mathbf{k}_e = \partial\mathbf{p}_e/\partial\mathbf{u}_e$ = element stiffness matrix in the global coordinate system. To isolate $\partial\mathbf{p}_e/\partial\theta|_{\mathbf{u}_e}$ in Eq. (13), it is necessary to establish a relationship between the derivatives $\partial\mathbf{u}/\partial\theta$ and $\partial\mathbf{u}_e/\partial\theta$. To this end, Eq. (4) is differentiated with respect to θ

$$\frac{\partial\mathbf{u}}{\partial\theta} = \mathbf{a}_r\frac{\partial\mathbf{u}_e}{\partial\theta} + \frac{\partial\mathbf{a}_r}{\partial\theta}\mathbf{u}_e \tag{14}$$

Then, Eqs. (13) and (14) are combined to give the following expression:

$$\mathbf{k}_e\frac{\partial\mathbf{u}_e}{\partial\theta} + \left.\frac{\partial\mathbf{p}_e}{\partial\theta}\right|_{\mathbf{u}_e} = \mathbf{a}_r^T\left(\mathbf{k}_l\mathbf{a}_r\frac{\partial\mathbf{u}_e}{\partial\theta} + \mathbf{k}_l\frac{\partial\mathbf{a}_r}{\partial\theta}\mathbf{u}_e + \left.\frac{\partial\mathbf{p}}{\partial\theta}\right|_{\mathbf{u}}\right) + \frac{\partial\mathbf{a}_r^T}{\partial\theta}\mathbf{p} \tag{15}$$

From the relationship between the element stiffness matrix in local and global coordinates, the first term on the left-hand side of Eq. (15) is equal to the first term in parentheses on the right-hand side of the equation. As a result, Eq. (15) reduces to the following expression for the conditional derivative of the element forces:

$$\left.\frac{\partial\mathbf{p}_e}{\partial\theta}\right|_{\mathbf{u}_e} = \mathbf{a}_r^T\mathbf{k}_l\frac{\partial\mathbf{a}_r}{\partial\theta}\mathbf{u}_e + \mathbf{a}_r^T\left.\frac{\partial\mathbf{p}}{\partial\theta}\right|_{\mathbf{u}} + \frac{\partial\mathbf{a}_r^T}{\partial\theta}\mathbf{p} \tag{16}$$

As described in the Appendix, the matrix $\partial\mathbf{a}_r/\partial\theta$ in Eq. (16) is equal to zero for all parameters in $\boldsymbol{\Theta}$ except those that refer to the nodal coordinates of the element. Eq. (16) defines the element contribution to the vector $\partial\mathbf{P}_r/\partial\theta|_{\mathbf{U}}$ in Eq. (3); however, it depends on the conditional derivative of the element forces in the local coordinate system, $\partial\mathbf{p}/\partial\theta|_{\mathbf{u}}$, which remains to be defined.

Differentiation of the Element Equilibrium and Compatibility Relationships

To determine the conditional derivative of the element forces in the local coordinate system, the element equilibrium relationship defined in Eq. (9) is differentiated by a procedure identical to that which led to Eq. (13)

$$\mathbf{k}_l\frac{\partial\mathbf{u}}{\partial\theta} + \left.\frac{\partial\mathbf{p}}{\partial\theta}\right|_{\mathbf{u}} = \mathbf{a}^T\left(\mathbf{k}\frac{\partial\mathbf{v}}{\partial\theta} + \left.\frac{\partial\mathbf{q}}{\partial\theta}\right|_{\mathbf{v}}\right) + \frac{\partial\mathbf{a}^T}{\partial\theta}\mathbf{q} \tag{17}$$

where $\mathbf{k}_l$ = element stiffness matrix in the local coordinate system, as defined in Eq. (11). From the second term on the right-hand side of Eq. (17), it is clear that the derivative of the element forces depends on the derivative of the element deformations, $\partial\mathbf{v}/\partial\theta$. This derivative is obtained by applying the chain rule of differentiation to $\mathbf{v} = \mathbf{v}(\mathbf{u}(\theta),\theta)$

$$\frac{\partial\mathbf{v}}{\partial\theta} = \mathbf{a}\frac{\partial\mathbf{u}}{\partial\theta} + \left.\frac{\partial\mathbf{v}}{\partial\theta}\right|_{\mathbf{u}} \tag{18}$$

where $\mathbf{a} = \partial\mathbf{v}/\partial\mathbf{u}$. The derivative of the element deformations in

Eq. (18) contains the conditional derivative, $\partial \mathbf{v}/\partial\theta|_{\mathbf{u}}$, because the removal of the rigid body displacement modes is a nonlinear function of the nodal displacements in the corotational theory. From the same derivation procedure as that employed for the gradient of the transformation from global to local coordinates, the insertion of Eq. (18) into Eq. (17) gives the following expression:

$$\mathbf{k}_l\frac{\partial \mathbf{u}}{\partial\theta} + \left.\frac{\partial \mathbf{p}}{\partial\theta}\right|_{\mathbf{u}} = \mathbf{a}^T\left(\mathbf{k}\mathbf{a}\frac{\partial \mathbf{u}}{\partial\theta} + \mathbf{k}\left.\frac{\partial \mathbf{v}}{\partial\theta}\right|_{\mathbf{u}} + \left.\frac{\partial \mathbf{a}^T}{\partial\theta}\right|_{\mathbf{v}}\right) + \frac{\partial \mathbf{a}^T}{\partial\theta}\mathbf{q} \quad (19)$$

It is possible to expand further the last term on the right-hand side of Eq. (19) by applying the chain rule to the differentiation of $\mathbf{a}^T$ with respect to θ

$$\frac{\partial \mathbf{a}^T}{\partial\theta} = \frac{\partial \mathbf{a}^T}{\partial \mathbf{u}}\frac{\partial \mathbf{u}}{\partial\theta} + \left.\frac{\partial \mathbf{a}^T}{\partial\theta}\right|_{\mathbf{u}} \quad (20)$$

With this derivative, Eq. (19) becomes

$$\mathbf{k}_l\frac{\partial \mathbf{u}}{\partial\theta} + \left.\frac{\partial \mathbf{p}}{\partial\theta}\right|_{\mathbf{u}} = \mathbf{a}^T\left(\mathbf{k}\mathbf{a}\frac{\partial \mathbf{u}}{\partial\theta} + \mathbf{k}\left.\frac{\partial \mathbf{v}}{\partial\theta}\right|_{\mathbf{u}} + \left.\frac{\partial \mathbf{q}}{\partial\theta}\right|_{\mathbf{v}}\right) + \left(\frac{\partial \mathbf{a}^T}{\partial \mathbf{u}}\frac{\partial \mathbf{u}}{\partial\theta} + \left.\frac{\partial \mathbf{a}^T}{\partial\theta}\right|_{\mathbf{u}}\right)\mathbf{q} \quad (21)$$

The sum of the two matrices on the right-hand side of Eq. (21) that multiply the vector $\partial\mathbf{u}/\partial\theta$ is equal to the element stiffness matrix $\mathbf{k}_l$ defined in Eq. (11). As a result, the first term on the left-hand side of Eq. (21) is equal to the sum of the two terms that involve the vector $\partial\mathbf{u}/\partial\theta$ on the right-hand side of the equation. On account of this equality, Eq. (21) reduces to

$$\left.\frac{\partial \mathbf{p}}{\partial\theta}\right|_{\mathbf{u}} = \mathbf{a}^T\mathbf{k}\left.\frac{\partial \mathbf{v}}{\partial\theta}\right|_{\mathbf{u}} + \left.\frac{\partial \mathbf{a}^T}{\partial\theta}\right|_{\mathbf{u}}\mathbf{q} + \mathbf{a}^T\left.\frac{\partial \mathbf{q}}{\partial\theta}\right|_{\mathbf{v}} \quad (22)$$

Eq. (22) represents the final expression for the conditional derivative of the element forces in the local coordinate system that is required for the computation of $\partial\mathbf{p}_e/\partial\theta|_{\mathbf{u}_e}$ according to Eq. (16). Three conditional derivatives contribute to $\partial\mathbf{p}/\partial\theta|_{\mathbf{u}}$, and the description of each follows.

The derivative in the first term of Eq. (22) is $\partial\mathbf{v}/\partial\theta|_{\mathbf{u}}$, the conditional derivative of the transformation that removes the rigid body displacement modes from the element displacement field. This vector represents the deformations that must be applied to the element to keep the nodal displacements fixed due to changes in the parameter θ. This derivative is obtained by differentiating the individual components of Eq. (6) with respect to θ

$$\frac{\partial \mathbf{v}}{\partial\theta} = \begin{bmatrix} \partial v_1/\partial\theta \\ \partial v_2/\partial\theta \\ \partial v_3/\partial\theta \end{bmatrix} = \begin{bmatrix} \partial L_n/\partial\theta - \partial L/\partial\theta \\ \partial u_3/\partial\theta - \partial\beta/\partial\theta \\ \partial u_6/\partial\theta - \partial\beta/\partial\theta \end{bmatrix} \quad (23)$$

As seen in Eq. (23), to determine $\partial v_1/\partial\theta$, it is necessary to compute the derivatives $\partial L/\partial\theta$ and $\partial L_n/\partial\theta$. The derivative $\partial L/\partial\theta$ is defined in the Appendix and the derivative of the deformed element length, L_n defined in Eq. (7), is

$$\frac{\partial L_n}{\partial\theta} = \cos\beta\left(\frac{\partial L}{\partial\theta} + \frac{\partial\Delta u_x}{\partial\theta}\right) + \sin\beta\frac{\partial\Delta u_y}{\partial\theta} \quad (24)$$

where $\partial\Delta u_x/\partial\theta = \partial u_4/\partial\theta - \partial u_1/\partial\theta$ and $\partial\Delta u_y/\partial\theta = \partial u_5/\partial\theta - \partial u_2/\partial\theta$ are the gradients of the relative displacements in the local coordinate system. The derivative of the rigid body rotation of the element chord is required to determine the derivatives $\partial v_2/\partial\theta$ and $\partial v_3/\partial\theta$ in Eq. (23). The differentiation of Eq. (8) with respect to θ gives

$$\frac{\partial\beta}{\partial\theta} = \frac{\cos\beta}{L_n}\frac{\partial\Delta u_y}{\partial\theta} - \frac{\sin\beta}{L_n}\left(\frac{\partial L}{\partial\theta} + \frac{\partial\Delta u_x}{\partial\theta}\right) \quad (25)$$

After inserting the derivatives of L, L_n, and β into Eq. (23), and collecting the terms that multiply the components of the vector $\partial\mathbf{u}/\partial\theta$, the derivative of the element deformations can be written as the sum of two terms

$$\frac{\partial \mathbf{v}}{\partial\theta} = \begin{bmatrix} -\cos\beta & -\sin\beta & 0 & \cos\beta & \sin\beta & 0 \\ -\sin\beta/L_n & \cos\beta/L_n & 1 & \sin\beta/L_n & -\cos\beta/L_n & 0 \\ -\sin\beta/L_n & \cos\beta/L_n & 0 & \sin\beta/L_n & -\cos\beta/L_n & 1 \end{bmatrix}\frac{\partial \mathbf{u}}{\partial\theta} + \begin{bmatrix} \cos\beta - 1 \\ \sin\beta/L_n \\ \sin\beta/L_n \end{bmatrix}\frac{\partial L}{\partial\theta} \quad (26)$$

In correspondence to Eq. (18), the matrix that multiplies $\partial\mathbf{u}/\partial\theta$ in the first term on the right-hand side of Eq. (26) is equal to $\mathbf{a}$, so the second term on the right-hand side of Eq. (26) must be equal to the conditional derivative of the element deformations

$$\left.\frac{\partial \mathbf{v}}{\partial\theta}\right|_{\mathbf{u}} = \begin{bmatrix} \cos\beta - 1 \\ \sin\beta/L_n \\ \sin\beta/L_n \end{bmatrix}\frac{\partial L}{\partial\theta} \quad (27)$$

Due to the common factor of $\partial L/\partial\theta$, the vector $\partial\mathbf{v}/\partial\theta|_{\mathbf{u}}$ is nonzero for only the parameters in $\boldsymbol{\Theta}$ that correspond to the nodal coordinates of the element.

The derivative in the second term of Eq. (22), $\partial\mathbf{a}/\partial\theta|_{\mathbf{u}}$, represents changes in the equilibrium transformation relationship due to variations in the parameter with the nodal displacements held fixed. With $\partial L_n/\partial\theta$ and $\partial\beta/\partial\theta$ defined, it is straightforward to determine the derivative of each component in the matrix $\mathbf{a}$ under the condition that $\partial\Delta u_x/\partial\theta$ and $\partial\Delta u_y/\partial\theta$ are zero:

$$\left.\frac{\partial \mathbf{a}}{\partial\theta}\right|_{\mathbf{u}} = \frac{1}{L_n^2}\begin{bmatrix} -L_n\sin^2\beta & L_n\cos\beta\sin\beta & 0 & L_n\sin^2\beta & -L_n\cos\beta\sin\beta & 0 \\ 2\cos\beta\sin\beta & 2\sin^2\beta - 1 & 0 & -2\cos\beta\sin\beta & 1 - 2\sin^2\beta & 0 \\ 2\cos\beta\sin\beta & 2\sin^2\beta - 1 & 0 & -2\cos\beta\sin\beta & 1 - 2\sin^2\beta & 0 \end{bmatrix}\frac{\partial L}{\partial\theta} \quad (28)$$

It is important to note that the matrix $\partial \mathbf{a}/\partial\theta|_{\mathbf{u}}$ is nonzero only when $\partial L/\partial\theta$ is nonzero, i.e., when the parameter θ corresponds to a nodal coordinate of the element.

Gradient Equations in the Basic System

The derivative in the third term of Eq. (22) is the vector $\partial \mathbf{q}/\partial\theta|_{\mathbf{v}}$, which is the contribution of the basic forces to the gradient of the element response. This derivative depends on the mathematical formulation of the equations of equilibrium and compatibility that govern the nonlinear material response of the beam-column element. The exact response gradient for the displacement- and force-based formulations of distributed inelasticity considering only uncertain material properties was outlined in the work of Scott et al. (2004). In this section, the response gradient for only the force-based formulation is extended to include uncertain nodal locations that define the orientation of the corotating reference frame within the global coordinate system. Derivations of response gradient equations for displacement-based and mixed formulations of distributed inelasticity are possible by a process identical to that presented herein, through the consistent differentiation of the equilibrium, compatibility, and constitutive equations that govern the element response in the basic system.

Regardless of the element formulation, the basic force vector, $\mathbf{q}\equiv\mathbf{q}(\mathbf{v}(\theta),\theta)$, is a nonlinear function of the element deformation vector, $\mathbf{v}$, and the parameter θ. To derive the element response gradient, it will be necessary to differentiate the basic force vector with respect to θ

$$\frac{\partial \mathbf{q}}{\partial\theta}=\mathbf{k}\frac{\partial \mathbf{v}}{\partial\theta}+\left.\frac{\partial \mathbf{q}}{\partial\theta}\right|_{\mathbf{v}} \tag{29}$$

where $\mathbf{k}=\partial\mathbf{q}/\partial\mathbf{v}$=element stiffness matrix in the basic system. A constitutive relationship at each cross section along the element gives the section forces, $\mathbf{s}\equiv\mathbf{s}(\mathbf{e}(\theta),\theta)$, as a nonlinear function of the section deformations, $\mathbf{e}$, and the parameter θ. The derivative of this section force-deformation relationship with respect to θ is

$$\frac{\partial \mathbf{s}}{\partial\theta}=\mathbf{k}_s\frac{\partial \mathbf{e}}{\partial\theta}+\left.\frac{\partial \mathbf{s}}{\partial\theta}\right|_{\mathbf{e}} \tag{30}$$

where $\mathbf{k}_s=\partial\mathbf{s}/\partial\mathbf{e}$=section stiffness matrix. Both the section forces, $\mathbf{s}$, and the conditional derivative of the section forces $\partial\mathbf{s}/\partial\theta|_{\mathbf{e}}$, are computed by either a stress-resultant plasticity model or a fiber representation of the cross section.

Force-Based Element Formulation

In the force-based beam-column formulation (Spacone et al. 1996; Neuenhofer and Filippou 1997), the equilibrium relationship is stated in strong form as

$$\mathbf{s}(x)=\mathbf{b}(x)\mathbf{q} \tag{31}$$

where the matrix $\mathbf{b}$ contains interpolation functions that relate section forces to basic forces, as determined from static equilibrium of the basic system in the undeformed configuration. Without loss of generality in the derivation of the response gradient, member loads are not included in Eq. (31), in which case the internal axial and shear forces are constant and the internal bending moment is linear at a section x along the element.

From the principle of virtual forces, compatibility between section deformations and element deformations is satisfied in integral form and evaluated numerically

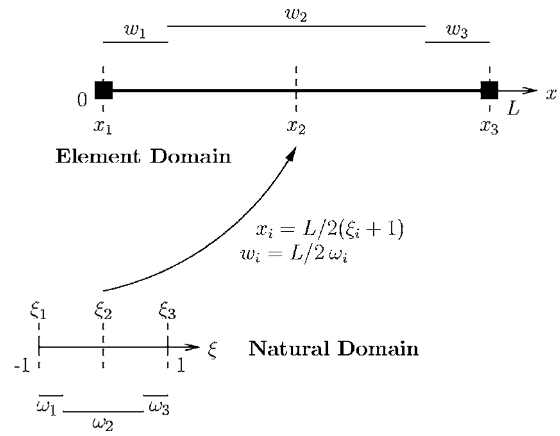

Fig. 4. The mapping of integration points from the natural domain to the element domain for three-point Gauss-Lobatto quadrature

$$\mathbf{v}=\sum_{i=1}^{N_p}\mathbf{b}^T(x_i)\mathbf{e}(x_i)w_i \tag{32}$$

over N_p discrete section locations. A common approach to evaluate Eq. (32) is Gauss-Lobatto quadrature, which specifies deterministic locations, ξ, and weights, ω, for the integration points in a natural domain $[-1,1]$ (Abramowitz and Stegun 1972). These locations are transformed to locations, x, and weights, w, in the element domain $[0,L]$ by the mapping shown schematically in Fig. 4 for three Gauss-Lobatto integration points (N_p=3).

In the force-based formulation, the element stiffness matrix in the basic system is determined by inversion of the element flexibility matrix, $\mathbf{k}=\mathbf{f}^{-1}$. The flexibility matrix is obtained by linearization of Eq. (32) with respect to the basic forces

$$\mathbf{f}=\frac{\partial \mathbf{v}}{\partial \mathbf{q}}=\sum_{i=1}^{N_p}\mathbf{b}^T(x_i)\mathbf{f}_s(x_i)\mathbf{b}(x_i)w_i \tag{33}$$

where $\mathbf{f}_s=\mathbf{k}_s^{-1}$ is the section flexibility matrix. After inversion of Eq. (33), the element stiffness is incorporated in the tangent stiffness matrix of the structure by the transformation in Eq. (11) and standard finite element assembly procedures.

Force-Based Element Response Sensitivity

The derivation of the response gradient for the force-based formulation begins with the differentiation of the equilibrium relationship in Eq. (31) taking into account variations in the basic and section force vectors and the force interpolation matrix

$$\frac{\partial \mathbf{s}}{\partial\theta}=\mathbf{b}\frac{\partial \mathbf{q}}{\partial\theta}+\frac{\partial \mathbf{b}}{\partial\theta}\mathbf{q} \tag{34}$$

After the insertion of the derivatives of the basic and section forces, from Eqs. (29) and (30), respectively, Eq. (34) expands to

$$\mathbf{k}_s\frac{\partial \mathbf{e}}{\partial\theta}+\left.\frac{\partial \mathbf{s}}{\partial\theta}\right|_{\mathbf{e}}=\mathbf{b}\left(\mathbf{k}\frac{\partial \mathbf{v}}{\partial\theta}+\left.\frac{\partial \mathbf{q}}{\partial\theta}\right|_{\mathbf{v}}\right)+\frac{\partial \mathbf{b}}{\partial\theta}\mathbf{q} \tag{35}$$

The conditional derivative of the basic forces, $\partial\mathbf{q}/\partial\theta|_{\mathbf{v}}$, cannot be determined from Eq. (35) because the force interpolation matrix, $\mathbf{b}$, in general, is not a square, invertible matrix and the vectors $\partial\mathbf{v}/\partial\theta$ and $\partial\mathbf{e}/\partial\theta$ are unknown. Adding to the difficulty is the fact that in the derivative of the element compatibility relationship

$$\frac{\partial \mathbf{v}}{\partial \theta} = \sum_{i=1}^{N_p} \left(\mathbf{b}^T \frac{\partial \mathbf{e}}{\partial \theta} + \frac{\partial \mathbf{b}^T}{\partial \theta} \mathbf{e} \right) w_i + \sum_{i=1}^{N_p} \mathbf{b}^T \mathbf{e} \frac{\partial w_i}{\partial \theta} \tag{36}$$

the derivative of the section deformations, $\partial \mathbf{e}/\partial \theta$, appears inside the summation, thus requiring further manipulation for the determination of $\partial \mathbf{q}/\partial \theta|_{\mathbf{v}}$ in the force-based formulation. The key to this derivation is to solve for $\partial \mathbf{e}/\partial \theta$ in Eq. (35)

$$\frac{\partial \mathbf{e}}{\partial \theta} = \mathbf{f}_s \mathbf{b} \mathbf{k} \frac{\partial \mathbf{v}}{\partial \theta} + \mathbf{f}_s \left(\mathbf{b} \frac{\partial \mathbf{q}}{\partial \theta} \bigg|_{\mathbf{v}} + \frac{\partial \mathbf{b}}{\partial \theta} \mathbf{q} - \frac{\partial \mathbf{s}}{\partial \theta} \bigg|_{\mathbf{e}} \right) \tag{37}$$

Eq. (37) expresses the derivative of the section deformations in terms of the conditional derivative of the basic force vector, $\partial \mathbf{q}/\partial \theta|_{\mathbf{v}}$, which is unknown at this point. The substitution of Eq. (37) into Eq. (36) yields

$$\frac{\partial \mathbf{v}}{\partial \theta} = \sum_{i=1}^{N_p} \mathbf{b}^T \mathbf{f}_s \left(\mathbf{b} \mathbf{k} \frac{\partial \mathbf{v}}{\partial \theta} + \mathbf{b} \frac{\partial \mathbf{q}}{\partial \theta} \bigg|_{\mathbf{v}} + \frac{\partial \mathbf{b}}{\partial \theta} \mathbf{q} - \frac{\partial \mathbf{s}}{\partial \theta} \bigg|_{\mathbf{e}} \right) w_i + \sum_{i=1}^{N_p} \left(\frac{\partial \mathbf{b}^T}{\partial \theta} \mathbf{e} w_i + \mathbf{b}^T \mathbf{e} \frac{\partial w_i}{\partial \theta} \right) \tag{38}$$

Then, the combination of terms involving the element flexibility matrix in Eq. (33) allows the previous equation to be expressed as

$$\frac{\partial \mathbf{v}}{\partial \theta} = \mathbf{f} \mathbf{k} \frac{\partial \mathbf{v}}{\partial \theta} + \mathbf{f} \frac{\partial \mathbf{q}}{\partial \theta} \bigg|_{\mathbf{v}} + \sum_{i=1}^{N_p} \mathbf{b}^T \mathbf{f}_s \left(\frac{\partial \mathbf{b}}{\partial \theta} \mathbf{q} - \frac{\partial \mathbf{s}}{\partial \theta} \bigg|_{\mathbf{e}} \right) w_i + \sum_{i=1}^{N_e} \left(\frac{\partial \mathbf{b}^T}{\partial \theta} \mathbf{e} w_i + \mathbf{b}^T \mathbf{e} \frac{\partial w_i}{\partial \theta} \right) \tag{39}$$

Using the identity $\mathbf{fk} = \mathbf{I}$, the term on the left-hand side of Eq. (39) and the first term on the right-hand side of the equation are equal. Then, the solution of Eq. (39) for the conditional derivative of the basic force vector gives

$$\frac{\partial \mathbf{q}}{\partial \theta} \bigg|_{\mathbf{v}} = \mathbf{k} \sum_{i=1}^{N_p} \mathbf{b}^T \mathbf{f}_s \left(\frac{\partial \mathbf{s}}{\partial \theta} \bigg|_{\mathbf{e}} - \frac{\partial \mathbf{b}}{\partial \theta} \mathbf{q} \right) w_i - \mathbf{k} \sum_{i=1}^{N_p} \left(\frac{\partial \mathbf{b}^T}{\partial \theta} \mathbf{e} w_i + \mathbf{b}^T \mathbf{e} \frac{\partial w_i}{\partial \theta} \right) \tag{40}$$

where $\partial \mathbf{s}/\partial \theta|_{\mathbf{e}}$ is determined from the gradient of the constitutive response at each section along the element. Two additional derivatives, $\partial \mathbf{b}/\partial \theta$ and $\partial w_i/\partial \theta$, are required to evaluate Eq. (40). First, for the common case when only axial force and bending moment are considered in the section response, it is straightforward to show that $\partial \mathbf{b}/\partial \theta = \mathbf{0}$ when the integration point locations are deterministic. Second, for deterministic integration weights defined in a natural domain, $\partial w_i/\partial \theta$ is found by differentiation of the mapping shown in Fig. 4, i.e., $\partial w_i/\partial \theta = (\omega_i/2)\partial L/\partial \theta$. As a result, when accounting for only axial force and bending moment in the response at sections with deterministic locations and weights, such as those dictated by Gauss-Lobatto quadrature, Eq. (40) reduces to

$$\frac{\partial \mathbf{q}}{\partial \theta} \bigg|_{\mathbf{v}} = \mathbf{k} \sum_{i=1}^{N_p} \mathbf{b}^T \mathbf{f}_s \frac{\partial \mathbf{s}}{\partial \theta} \bigg|_{\mathbf{e}} w_i - \mathbf{k} \mathbf{v} \frac{1}{L} \frac{\partial L}{\partial \theta} \tag{41}$$

There are two situations where Eq. (41) does not apply and the general form of Eq. (40) must be used to compute the conditional derivative of the basic forces. First, if the integration point locations and weights depend on an uncertain parameter, e.g., when θ corresponds to a prescribed plastic hinge length (Scott and Fenves 2006), $\partial \mathbf{b}/\partial \theta$ and $\partial w_i/\partial \theta$ will be nonzero. These derivatives must be computed consistent with the element integration method. Second, if shear force is considered in the section response, $\partial \mathbf{b}/\partial \theta$ will be nonzero when $\partial L/\partial \theta$ is nonzero, regardless of the element integration approach. This is due to the dependence of the internal shear force on the element length through equilibrium with the end moments of the basic system.

At this point, each term required to determine the conditional derivative of the basic forces in the force-based formulation via Eq. (40) is defined and computable. In the two-phase process for path-dependent response gradient computations, the conditional derivative of the basic forces is determined from Eq. (40) for eventual incorporation in Eq. (16) and assembly into the right-hand side of Eq. (3) during phase one. Then, in phase two, the derivative of the element deformation vector is computed from the nodal response gradient according to Eq. (23) and subsequently used in Eq. (37) to determine the derivative of the section deformations in order to update the gradient of the internal history variables for the element.

Numerical Examples

The proposed equations to compute the response gradient for the force-based formulation within the corotational frame of reference for large displacement structural analysis have been implemented in the finite element framework OpenSees (McKenna et al. 2000). The following numerical examples validate the exact response gradient equations and demonstrate the accuracy and efficiency of the DDM approach.

For a load-control solution strategy, where the structural displacements are determined from an applied load that is held constant at each iteration during a time step of the simulation, the exact gradient of the structural response, $\partial \mathbf{U}/\partial \theta$, can be verified by comparison with the finite difference approximation

$$\lim_{\varepsilon \to 0} \frac{\mathbf{U}(\theta + \varepsilon\theta) - \mathbf{U}(\theta)}{\varepsilon\theta} = \frac{\partial \mathbf{U}}{\partial \theta} \tag{42}$$

In the limit as the perturbation, ε, approaches zero, the finite difference approximation should converge to the exact gradient. Implicit in Eq. (42) is the assumption that the applied load vector, $\mathbf{P}_f$, follows the same path over the course of the simulation for every perturbation of the parameter.

Lee's Frame

The response gradient for the case of nonlinear geometric behavior of a linear elastic structure is verified in this example. The structural model is Lee's frame, which is a standard test problem for geometric nonlinear structural analysis (Cichon 1984; Park and Lee 1996). Lee's frame is a two-member L-shaped structure with an eccentric load applied on the beam, as shown in the inset of Fig. 5. Each member has a length of $L = 120$ cm, elastic modulus of $E = 720$ MPa and a 1.0 cm by 3.0 cm rectangular cross section. Ten force-based beam-column elements with linear-elastic deformations in the basic system are used for each member of Lee's frame in order to represent its complex nonlinear geometric behavior. De Souza (2000) showed that a single force-based element per member can represent accurately the response of Lee's frame by taking into account moderate section deformations in the element formulation.

The computed response is shown in Fig. 5 as a plot of the applied load versus the displacement at the same degree of

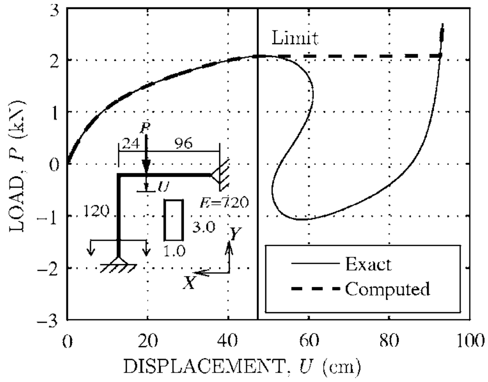

Fig. 5. Load-displacement response of Lee's frame computed with a load-control incremental solution strategy

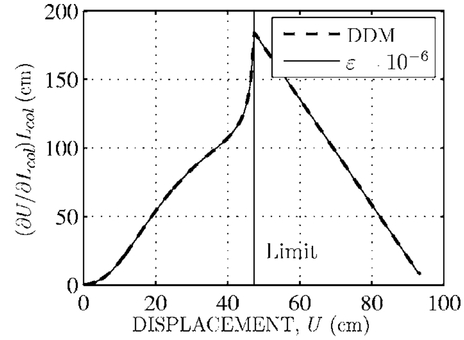

Fig. 7. Response gradient of Lee's frame computed with respect to the column length by the direct differentiation and finite-difference methods

freedom as the applied load. To compute the load-displacement response in OpenSees, a load-controlled incremental solution strategy is used with constant load steps of $\Delta P = 0.005$ kN up to a total load of $P = 2.7$ kN. As the computed solution is load controlled, it does not capture the snap-through and snap-back response shown for the exact solution and instead it jumps through the region of geometric instability. As the intent of this example is to demonstrate the sensitivity of the frame response to uncertain parameters as it approaches the limit point, the load-controlled solution is sufficient.

The gradient of the frame response is computed with respect to three potentially uncertain parameters: The elastic modulus, E; the length of the column member, L_{col}; and the length of the beam member, L_{beam}. The response sensitivity with respect to L_{col} and L_{beam} is based on treating the Y coordinate of the column support and the X coordinate of the beam support, respectively, as uncertain while all other nodal locations in the finite element mesh are deterministic. The response gradient computed by the DDM for each parameter is shown in Figs. 6–8, where the DDM results are compared with finite difference computations for a small parameter perturbation, $\varepsilon = 10^{-6}$. As expected, the response becomes extremely sensitive to each parameter as the frame approaches the limit point at a displacement of about 47 cm. Similar to the actual response, the response gradient jumps through the region of geometric instability. As indicated by the negative response gradient values in Fig. 6, the elastic modulus, E, is a resistance variable, i.e., an increase in E will cause a decrease in U. On the other hand, as seen in Figs. 7 and 8, the lengths of the frame members are load variables as an increase in these parameters will cause an increase in displacement. For each of the parameters considered, the DDM computations match those computed by finite differences, thereby validating the DDM response gradient equations and implementation.

Slender Steel Cantilever

In this example, the response gradient for nonlinear geometric response combined with path-dependent nonlinear material response in the basic system according to the force-based formulation is verified. The structural model is a steel cantilever of length $L = 5.0$ m with cross-sectional dimensions of $b = 0.1$ m width and $d = 0.5$ m depth. A single force-based element with five Gauss-Lobatto integration points represents the nonlinear material response of the cantilever. The cantilever is loaded through one sinusoidal cycle with a peak magnitude of $5M_y/L$ using the load-control solution strategy. Thus, the finite difference approximation of Eq. (42) is appropriate for the verification of the DDM gradient computations. The elastic modulus for the cantilever is $E = 2.0 \times 10^5$ MPa and the yield stress is $\sigma_y = 410$ MPa. A bilinear moment-curvature relationship with a hardening ratio of 7% represents the section behavior at each integration point. The axial behavior is assumed linear elastic.

The load-displacement response for the steel cantilever is shown in Fig. 9 for the corotational large displacement theory along with the case of linear geometry, i.e., small displacement theory. The cantilever response is similar for both cases before

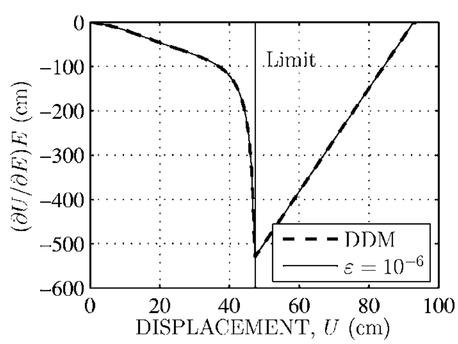

Fig. 6. Response gradient of Lee's frame computed with respect to the material modulus of elasticity by the direct differentiation and finite-difference methods

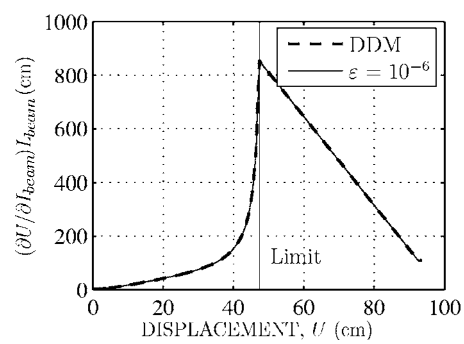

Fig. 8. Response gradient of Lee's frame computed with respect to the beam length by the direct differentiation and finite-difference methods

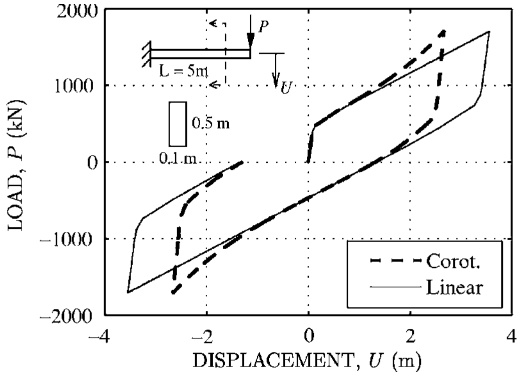

Fig. 9. Load-displacement response of slender steel cantilever using one force-based element considering small displacement linear geometry and large displacement nonlinear geometry according to the corotational theory

and after a plastic hinge is formed; however, as the cantilever displacement exceeds 20% drift, geometric effects start to dominate the response.

The DDM response gradient with respect to the element length, L, is validated by comparison with finite difference computations, as shown in Fig. 10(a). As the parameter perturbation decreases, the finite differences converge to the DDM gradient, as expected according to Eq. (42). It is noted that the cantilever length is a load variable since the cantilever displacement will increase with an increase in L. The response gradient is compared with the gradient for the case of small displacements in Fig. 10(b). At the peak load (pseudotime, $t=1$), the cantilever response is about twice as sensitive to the length under the as-

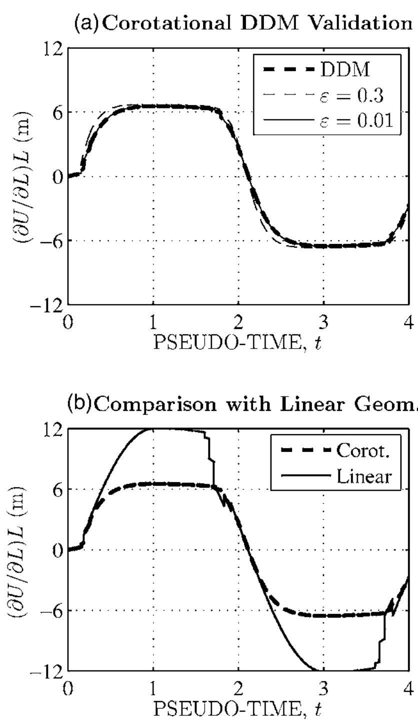

Fig. 10. Response gradient of slender steel cantilever computed with respect to the cantilever length: (a) validation of the corotational DDM computation by finite differences; (b) comparison of corotational gradient with that for linear geometry

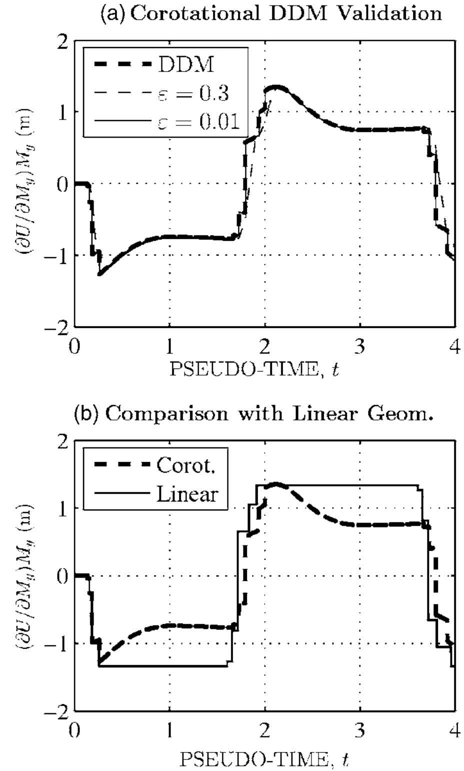

Fig. 11. Response gradient of slender steel cantilever computed with respect to the yield moment: (a) validation of the corotational DDM computation by finite differences; (b) comparison of corotational gradient with that for linear geometry

sumption of small displacements compared to the case for large displacements. Thus, the response sensitivity computation reflects the fact that the assumption of linear geometry can lead to significant overestimation of structural displacements under a given loading.

To examine the DDM computations for the case of a material parameter, the cantilever response gradient is computed with respect to the section yield moment, M_y. As shown in Fig. 11(a), the gradients computed by the finite difference method converge to that computed by the DDM. The transition between elastic and plastic states at each section causes a discrete jump in the response gradient (Conte et al. 1999). Further, Fig. 11(a) indicates that the section yield moment is a resistance variable because an increase in M_y will decrease the displacement of the cantilever. A comparison of the corotational response sensitivity with that for linear geometry is shown in Fig. 11(b), where it is seen that the sensitivity remains constant as the beam strain hardens under linear geometry, while there is a reduction in the sensitivity as strain hardening takes place under large displacements. This indicates that the tip displacement becomes less sensitive to changes in the material parameter as the nonlinear geometric response begins to dominate the cantilever response.

Conclusions

The proposed response gradient formulation makes possible the reliable and accurate assessment of the effect of uncertain material and geometric parameters in the large displacement analysis of frames. To account for geometric nonlinearity due to large displacements, each component in the corotational theory was differentiated in a consistent manner. This includes the transformation between the global and local coordinate systems, removal of the rigid body displacement modes, and equilibrium between basic and local forces for a beam-column element. Within the corotating frame of reference, the governing equations of compatibility and equilibrium for the force-based formulation were differentiated to account for nonlinear material response of the element.

The derivations and numerical examples presented in this paper focused on the force-based formulation for nonlinear material response; however, similar derivations are possible for displacement-based and mixed formulations. The numerical examples validated the proposed response gradient equations in large displacement structural analysis. Consequently, this paper represents an important step toward the use of material nonlinear beam formulations within the framework of the large displacement corotational theory for gradient-based applications in structural engineering.

Appendix. Gradient of the Global to Local Transformation

The transformation matrix, $\mathbf{a}_r$, that describes the transformation of element displacements and forces between global and local coordinates is block diagonal with 2×2 Givens rotation matrices for the element projections in the global coordinate system. The projection onto the global X axis, $\cos\alpha=\Delta X/L$, is determined from the difference between the global X-coordinates of the element, $\Delta X=X_J-X_I$. Similarly, the global Y-axis projection, $\sin\alpha=\Delta Y/L$, is determined from the Y coordinates as $\Delta Y=Y_J-Y_I$.

The derivatives of $\cos\alpha$ and $\sin\alpha$, required to populate the matrix $\partial\mathbf{a}_r/\partial\theta$ in Eq. (16), are obtained by direct differentiation of the element projections

$$\frac{\partial(\cos\alpha)}{\partial\theta}=\frac{1}{L}\frac{\partial\Delta X}{\partial\theta}-\frac{\cos\alpha}{L}\frac{\partial L}{\partial\theta} \tag{43}$$

$$\frac{\partial(\sin\alpha)}{\partial\theta}=\frac{1}{L}\frac{\partial\Delta Y}{\partial\theta}-\frac{\sin\alpha}{L}\frac{\partial L}{\partial\theta} \tag{44}$$

The derivative $\partial\Delta X/\partial\theta$ will be zero if θ does not represent an X coordinate of one of the element nodes, whereas it will be equal to 1 if θ represents the coordinate X_J and equal to -1 if θ represents X_I. Similarly, $\partial\Delta Y/\partial\theta$ will be equal to zero or ± 1.

The remaining term in Eqs. (43) and (44) is $\partial L/\partial\theta$, the derivative of the undeformed element length, $L=\sqrt{(\Delta X)^2+(\Delta Y)^2}$. It is straightforward to show the derivative of L with respect to θ is

$$\frac{\partial L}{\partial\theta}=\cos\alpha\frac{\partial\Delta X}{\partial\theta}+\sin\alpha\frac{\partial\Delta Y}{\partial\theta} \tag{45}$$

As a result, $\partial L/\partial\theta$ can only take on values of zero, $\pm\cos\alpha$, or $\pm\sin\alpha$. From the structure of Eqs. (43)–(45), the matrix $\partial\mathbf{a}_r/\partial\theta$ will be nonzero for only the parameters in $\boldsymbol{\Theta}$ that represent a nodal coordinate of the element.

Notation

The following symbols are used in this paper:

$\mathbf{a}$ = equilibrium transformation matrix;
$\mathbf{a}_r$ = transformation matrix between the global and local coordinate systems;
$\mathbf{b}$ = section force interpolation matrix;
$\mathbf{e}$ = section deformation vector;
$\mathbf{f}$ = element flexibility matrix;
$\mathbf{f}_s$ = section flexibility matrix;
K_T = tangent stiffness matrix for the structure;
$\mathbf{k}$ = element stiffness matrix in the basic system;
$\mathbf{k}_e$ = element stiffness matrix in the global coordinate system;
$\mathbf{k}_l$ = element stiffness matrix in the local coordinate system;
$\mathbf{k}_s$ = section stiffness matrix;
$\mathbf{P}_f$ = external load vector for the structure;
$\mathbf{P}_r$ = resisting force vector for the structure;
$\mathbf{p}$ = element force vector in the local coordinate system;
$\mathbf{p}_e$ = element force vector in the global coordinate system;
$\mathbf{q}$ = element force vector in the basic system;
$\mathbf{s}$ = section force vector;
$\mathbf{v}$ = element deformation vector;
$\mathbf{U}$ = nodal displacement vector for the structure;
$\mathbf{u}$ = element displacement vector in the local coordinate system;
$\mathbf{u}_e$ = element displacement vector in the global coordinate system;
ξ, x = integration point location in natural, element domain;
ω, w = integration weight in natural, element domain;
$\boldsymbol{\Theta}$ = vector of uncertain parameters; and
θ = uncertain parameter.

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