

# Development of Bridge Rating Applications Using OpenSees and Tcl

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**Abstract:** A bridge rating application is developed using the Tool Command Language (Tcl) scripting language in conjunction with the OpenSees finite-element software framework. Fully programmable and string based, Tcl is ideal for implementing live load analysis through scripts and experimenting with emergent bridge rating methodologies. Since Tcl is an interpreted language, the application also has the important advantage that new bridge capacity models and rating factor calculations can be implemented quickly by a user without compiling source code. The rating application is demonstrated on a conventionally reinforced concrete bridge girder. The use of OpenSees for finite-element analysis makes the rating application readily extensible to nonlinear structural response and advanced structural reliability methods for a variety of a bridge types and structural components.

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**CE Database subject headings:** Bridge maintenance; Computer aided simulation; Computer programming; Live loads; Ratings; Structural analysis.

## Introduction

Rating is the primary tool for evaluating the suitability of highway bridges for continued service. To rate a bridge component, the load effect demands from truck configurations are computed by structural analysis, then compared with an estimated capacity of the component. Bridge rating requires a significant amount of time and produces a large amount of data; fortunately, it has been automated in several software applications. For a closed software model where only a few developers have access to source code, it is often difficult for a user to incorporate new simulation models for bridge components, to modify the capacity and rating factor calculations to keep up with changing code provisions, to experiment with emerging rating methodologies such as those based on structural reliability (Akgül and Frangopol 2003; Higgins et al. 2005), or to extend the software to system-level models of bridge network performance (Liu and Frangopol 2006). The Alternate Route Project (Brice 1999) uses an open source model to develop bridge design and analysis applications. Raymond (2001) de-

scribes the advantages of open source software development over closed source approaches.

The open source approach has also been adopted in the development of the software framework OpenSees, which consists of a set of cooperating modules that can be used to construct applications for structural and geotechnical engineering (McKenna et al. 2000). OpenSees is based on the finite-element method (Cook et al. 1989; Bathe 1996; Zienkiewicz and Taylor 2000), which is the most versatile approach to computing the response of structural systems subjected to general loadings. OpenSees is designed in a modular fashion to support the finite-element method with loose coupling of analysis and model building components (McKenna 1997). This allows users and developers in different disciplines, including engineering, computer science, and numerical analysis, to modify or implement specific modules with relatively little dependence on other modules. Developers do not need to know everything that is in the framework, allowing them to make improvements or create applications in areas of their expertise. Furthermore, modules can be optimized to take advantage of computing hardware, communication, and visualization without the changes propagating throughout the entire system.

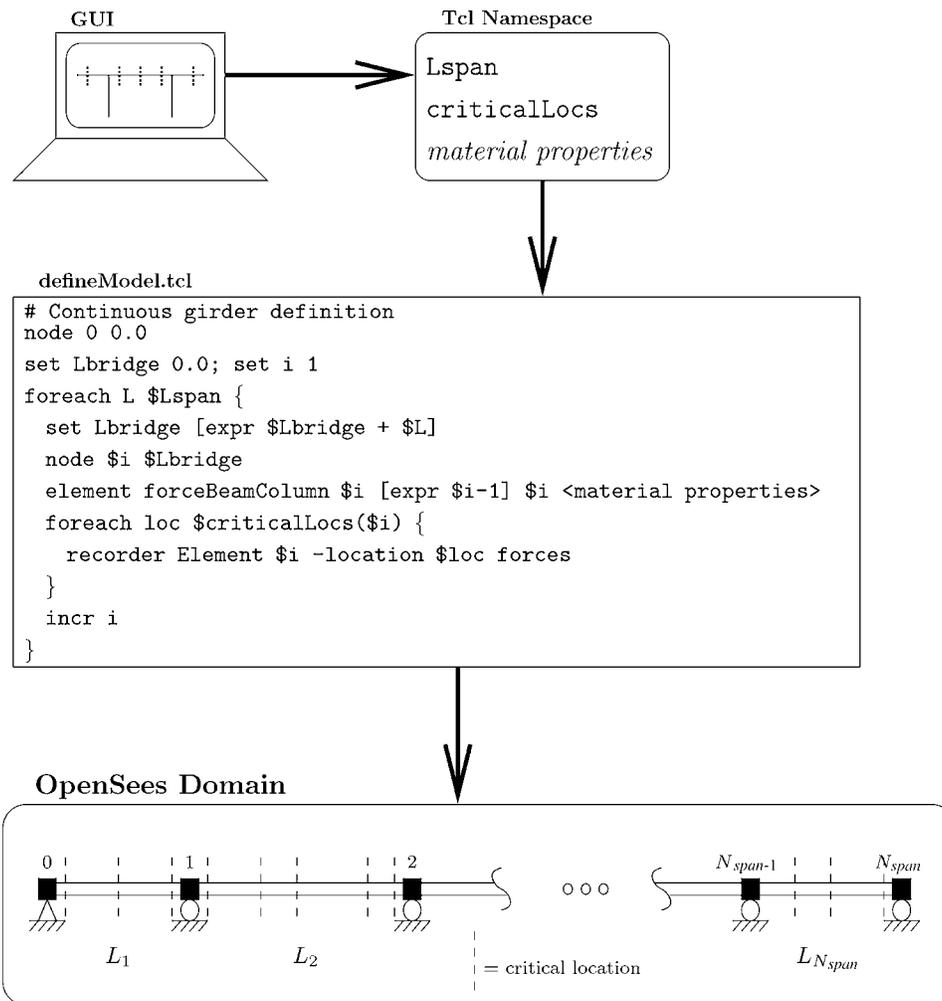
Most users of OpenSees build models and conduct analyses via the string-based Tool Command Language (Tcl) (Ousterhout 1994). Tcl is fully programmable with the control structures, variable substitutions, and procedures that are necessary to automate routine operations using scripts (Welch 2000). The aim of Tcl is to serve as a glue language that assembles software building blocks into customized applications. This is accomplished by allowing developers to extend the Tcl interpreter with commands that suit the needs of an application. In the case of OpenSees, the Tcl interpreter is extended with commands to define the nodes, boundary conditions, elements, loads, and solution strategies of a finite-element analysis (Mazzoni et al. 2006). There are many advantages to using a fully programmable, interpreted language such as Tcl for defining models and solution methods in OpenSees, including the ability to conduct parameter studies, to provide network access to data and storage, to control hybrid simulations, and to communicate with graphical user interfaces (Peng and Law

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**Fig. 1.** Tcl code to create nodes and finite-elements that represent continuous bridge girder and to record analysis results for each critical location

2002; Takahashi and Fenves 2005; Schellenberg and Mahin 2006; Hutchinson et al. 2007).

The objective of this paper is to outline how Tcl is used as a glue to construct bridge rating applications from finite-element analysis building blocks in the OpenSees framework. The presentation begins with scripts to define bridge models and compute demands by moving load finite-element analysis in OpenSees. The subsequent sections present procedures to compute the moment and shear capacity of conventionally reinforced concrete girders and to implement specification-based rating equations. An example of bridge rating for girder moment and shear is shown, then the paper concludes with future extensions of the bridge rating application.

### Scripts for Live Load Bridge Analysis

The typical approach to compute live load demands in continuous bridge girders is to perform static, two-dimensional analyses. Distribution factors approximate three-dimensional effects of load transfer through the bridge deck and an impact factor accounts for dynamic effects of vehicle loading. Axle weights are represented as point loads and a uniformly distributed load represents self-weight of the wearing surface and structural components. The demands due to dead and live loads are recorded at critical girder locations that coincide with changes in reinforcing details and

transitions in builtup sections, as determined from design drawings. Other locations of interest may be chosen to reflect field inspection.

In the following sections, Tcl scripts for model building and moving load analysis in OpenSees are presented. Qualitative analysis of moving loads based on the Müller-Breslau principle and superposition of influence functions is cumbersome to incorporate in a general finite-element setting. Scripting languages such as Tcl are suited to quantitative analysis of moving loads where the position and combination of loads that will produce the maximum demands are determined by repeated structural analyses. The procedures for model building and load positioning demonstrate Tcl syntax and lay the foundation for the bridge rating application presented herein.

### Specification of Finite-Element Model

To represent a continuous bridge girder in OpenSees, a mesh of nodes and beam elements is generated by the Tcl script `defineModel.tcl` shown in Fig. 1. The script invokes the `node` and `element` commands, added to the Tcl interpreter by OpenSees, to build the finite-element model using two arrays, `Lspan` and `criticalLocs`, added to the Tcl namespace via a graphical user interface. The `Lspan` array contains the length of each span while the `criticalLocs` array contains the critical locations along each span. To iterate over spans, the `foreach` loop is preferred over a `for`

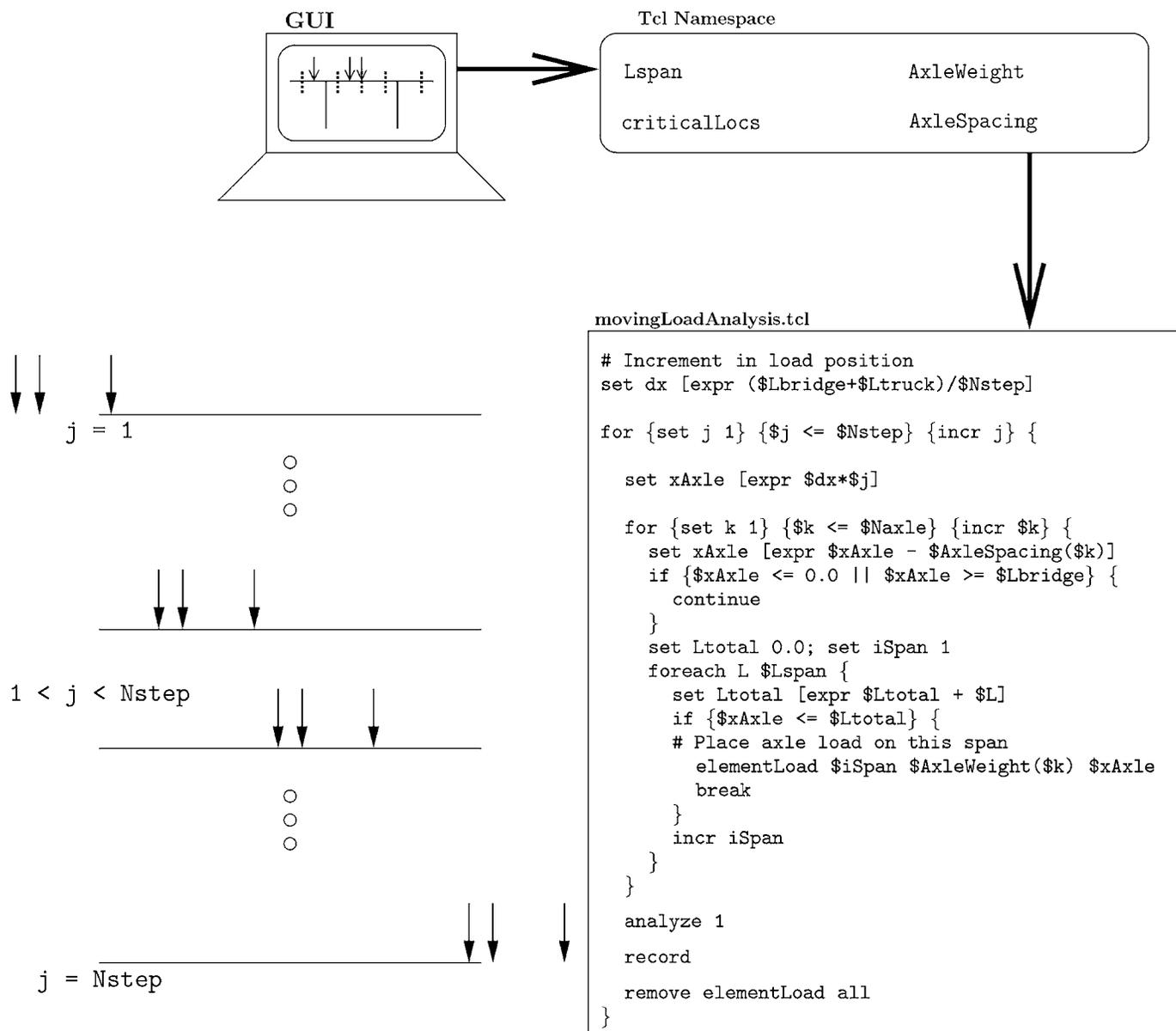


Fig. 2. Tcl code to move multi-axle vehicle loading across continuous bridge girder

loop since the indices of a Tcl array can be any string value, not just integers. The `recorder` command is issued at each critical location so that the internal forces will be logged at each step of the live load analysis. Although omitted from Fig. 1 for brevity, the section and material properties for each critical location must be defined in the Tcl namespace in order to determine the distribution of forces along the girder during the analysis.

### Bridge Analysis for Moving Loads

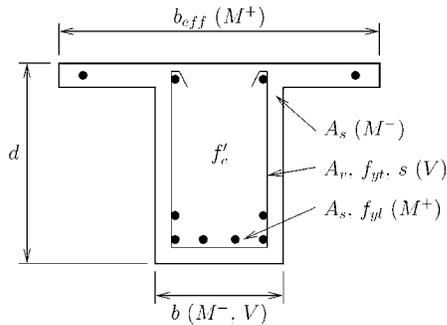
A Tcl script that moves axle loads across the bridge model is shown in Fig. 2. In addition to the variables created in the model definition, the arrays `AxleWeight` and `AxleSpacing` containing the magnitude and relative spacing of axle weights are required to conduct the moving load analysis. These arrays can be populated by a user via a script or graphical user interface, or queried from a database.

As shown in Fig. 2, a short control sequence determines where to apply each axle load. After all axle loads are placed on the

bridge, the `analyze` command invokes the structural analysis procedures, and finite-element solution methods of the OpenSees framework. Once the demand is recorded at each critical location, all axle loads are removed from the bridge. Then these loads are moved to new locations based on a constant increment and the analysis is repeated. This process continues until all loads move off the bridge.

### Tcl Procedures for Girder Capacity Specification

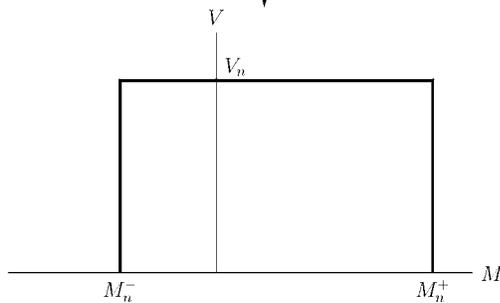
This section describes the implementation of a Tcl procedure to compute the moment and shear capacity of reinforced concrete bridge girders. Multiple moment-shear capacity formats exist, ranging from the use of design equations specified by ACI (2005) to those based on the modified compression field theory (MCFT) (Vecchio and Collins 1988). Regardless of the format, the capacity calculation can be encapsulated by a Tcl procedure. The ACI



```

proc ACI_318_05 {fc fyl fyt AsPos AsNeg d b beff s} {
  set aPos [expr $AsPos*$fyl/(0.85*$fc*$beff)]
  set capacity(MnPos) [expr $AsPos*$fyl*($d-$aPos/2)]
  set aNeg [expr $AsNeg*$fyl/(0.85*$fc*$b)]
  set capacity(MnNeg) [expr $AsNeg*$fyl*($d-$aNeg/2)]
  set Vc [expr sqrt($fc)/6*$bw*$dNeg]
  set Vs [expr $Av*$fyt*$dNeg/$s]
  set capacity(Vn) [expr $Vc+$Vs]
  return $capacity
}

```



**Fig. 3.** Calculation of moment and shear capacities for girder section using ACI specifications implemented in Tcl procedure

318-05 design equations for moment and shear capacity of reinforced concrete beams demonstrate the approach. The nominal moment capacity is a function of the longitudinal steel reinforcement and the section geometry

$$M_n = A_s f_{yt} \left( d - \frac{a}{2} \right) \quad (1)$$

The depth of the equivalent stress block is

$$a = \frac{A_s f_{yt}}{0.85 f'_c b} \quad (2)$$

Eq. (1) for  $M_n$  computes positive and negative moment capacities using the section dimensions shown in Fig. 3 for  $M^+$  and  $M^-$ . The nominal shear capacity is the sum of contributions from the concrete and transverse steel reinforcement

```

proc LRFRrating {phi gammaDC gammaDW gammaL R DC DW LL} {
  # phi - resistance factor
  # gammaDC - load factor for dead weight of bridge components
  # gammaDW - load factor for dead weight of bridge wearing surface
  # gammaLL - load factor for live loads
  # R - nominal resistance to force effect
  # DC - force effect due to dead weight of bridge components
  # DW - force effect due to dead weight of bridge wearing surface
  # LL - force effect due to live load including impact

  set capacity [expr $phi*$R - $gammaDC*$DC - $gammaDW*$DW]

  set demand [expr $gammaL*$LL]

  return [expr $capacity/$demand]
}

```

**Fig. 4.** Tcl procedure to compute rating factor based on AASHTO-LRFR specifications

$$V_n = V_c + V_s = \frac{\sqrt{f'_c}}{6} b d + \frac{A_v f_{yt} d}{s} \quad (\text{MPa}) \quad (3)$$

These design equations are implemented in the Tcl procedure ACI\_318\_05 shown in Fig. 3. Input for the procedure consists of the material properties, dimensions, and reinforcing details of a reinforced concrete section (assuming consistent units). The procedure returns an array of nominal values for positive and negative moment capacities and shear capacity.

### Tcl Commands for Rating Factor Calculations

With the demands and capacity known at each critical girder location it is possible to compute a rating factor, or ratio of capacity to demand. A rating factor that is less than 1.0 indicates the bridge component may require repair or strengthening. Several approaches are available to determine rating factors for bridge girders considering flexure, shear, or some combination of the two. Using procedures and mathematical expressions in Tcl, specification-based rating factors considering individual force effects are straightforward to implement. The most basic rating format is given by Eq. (6-1) in AASHTO (2003b)

$$RF = \frac{\phi R - \gamma_{DC} DC - \gamma_{DW} DW}{\gamma_{LL} LL_{IM}} \quad (4)$$

where  $R$  = nominal resistance to the force effect, as determined from design drawings. Permanent load effects, DC and DW, due to the weight of structural components and wearing surface, respectively, and live load  $LL_{IM}$  accounting for dynamic load effects (via an impact factor) are obtained from finite analysis in OpenSees. The resistance factor,  $\phi$ , and load factors,  $\gamma$ , are tabulated in the LRFR manual for a variety of force effects, bridge types, and service states. Further details on this rating format are given by Minervino et al. (2004). A Tcl procedure to compute rating factors for individual force effects according to Eq. (4) is shown in Fig. 4.

### Overview of Rating Application

A global picture of the rating application is shown in Fig. 5 as an interaction between a graphical user interface and variables and scripts executed by the Tcl interpreter. After setting variables for the bridge geometry, critical locations, and material properties,

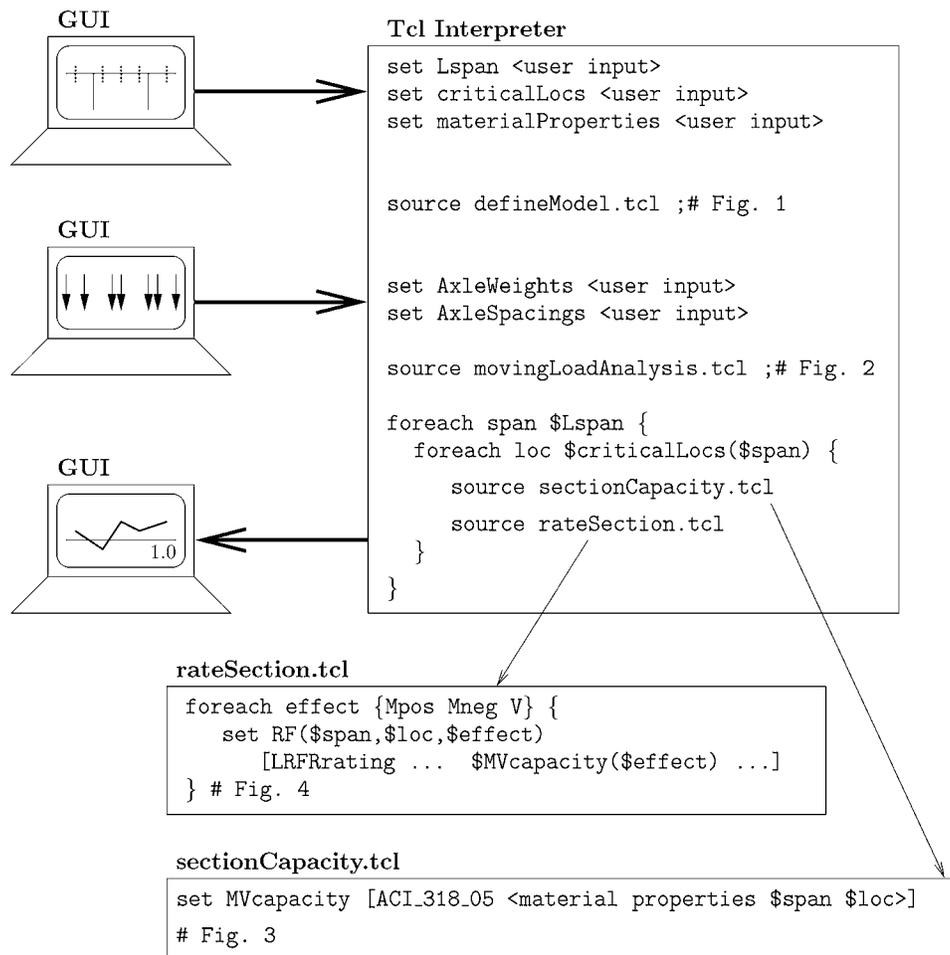


Fig. 5. Overview of variable definitions, script executions, and procedure calls necessary for bridge rating application

the finite-element model is defined by evaluating the `defineModel.tcl` script shown in Fig. 1. The model is then analyzed by executing the `movingLoadAnalysis.tcl` script in Fig. 2 for the user-specified load configuration of axle weights and spacings. The application then iterates over all critical locations, computing the moment-shear capacity (via a script that calls the `ACI_318_05` procedure in Fig. 3) and calculating rating factors by invoking a script that calls the `LRFRrating` procedure in Fig. 4 for all force effects of interest. The execution of capacity and rating calculations in separate scripts allows a user to easily change their implementation without affecting the sequence of interpreted commands shown in Fig. 5.

The use of Tcl makes the rating application an open environment that can incorporate advanced rating methodologies based on risk and reliability (Stewart et al. 2001; Akgül and Frangopol 2003, 2004). Furthermore, the application allows developers of emergent rating methodologies to utilize the advanced finite-element reliability analysis modules that are available in OpenSees (Haukaas and Der Kiureghian 2007) in order to account for a wide range of uncertain input parameters in the bridge rating.

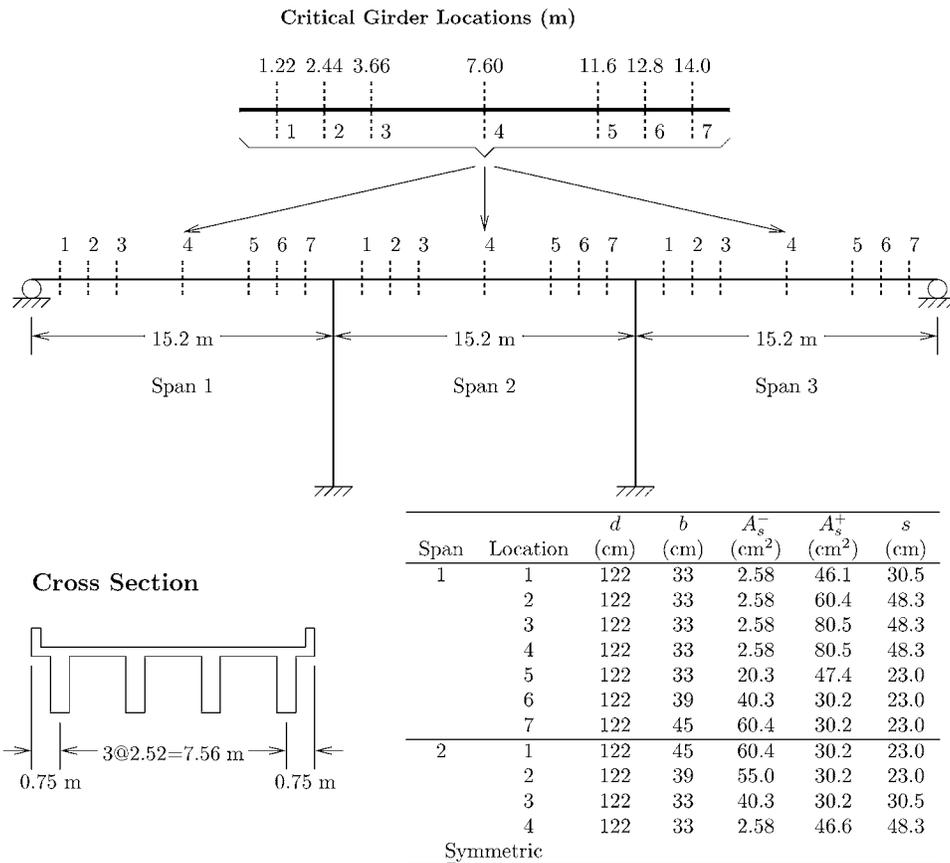
### Example Bridge Girder Rating Application

An interior girder of the McKenzie River Bridge, a reinforced concrete deck girder (RCDG) bridge located on Interstate-5 just north of Eugene, Ore. serves as an example for the OpenSees/Tcl

bridge rating application. An idealized model of the girder is shown in Fig. 6 as three spans of equal length (15.2 m). For demonstration purposes, critical girder locations are assumed at midspan and  $d$ ,  $2d$ , and  $3d$  away from the supports, where  $d=122$  cm=girder depth. Distribution factors for moment and shear of the interior girder are calculated as 0.854 and 0.884, respectively, using a Tcl procedure that implements distribution factors for reinforced concrete deck girder bridges, as prescribed in Section 4.6.2 of AASHTO (2003a). An impact factor of 1.33 is used in the analysis. The combined weight of permanent loads (structural components and wearing surface) is 26.3 kN/m.

The girder response remains in the linear-elastic range, as verified by field tests (Potisuk and Higgins 2007). The changes in girder width shown in the table of Fig. 6 are accounted for explicitly by numerical integration in force-based finite-elements (Spacone et al. 1996; Neuenhofer and Filippou 1997). The integration points coincide with the user-specified critical girder locations without a significant loss of numerical accuracy (Kidarsa 2006). The use of force-based elements for the OpenSees analysis also facilitates the extension to nonlinear effects of girder moment-shear interaction (Ranzo and Petrangeli 1998; Marini and Spacone 2006) due to overload conditions.

The rating factors for moment and shear of an interior girder under the OR-STP-5B load configuration (Fig. 7), representative of single-trip permit (STP) vehicles in Oregon, are shown in Fig. 8. The finite-element model is symmetric as are the loading and the location of the critical section, thus only half of the bridge is



**Fig. 6.** Span numbers, lengths, and critical locations for McKenzie River Bridge

shown in Fig. 8. The OR-STP-5B loading moves across the bridge forward and backward in 500 load increments to ensure that the extreme load effects are captured. The example uses resistance factors,  $\phi$ , of 0.9 and 0.75 to rate for moment and shear, respectively, and factors  $\gamma_L = 1.5$  for live loads and  $\gamma_{DC} = \gamma_{DW} = 1.2$  for permanent loads. As shown in Fig. 8, the girder does not rate sufficiently for shear near the bridge supports nor for positive moment at 11.6 m from the approaches. This indicates that these portions of the bridge require strengthening to support the weight configuration or the bridge may require load posting.

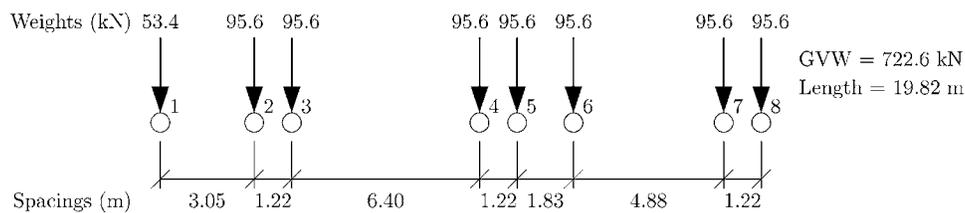
The load histories at two critical girder locations in Fig. 9 compare the demands, including load factors and distribution and impact factors, computed for all steps of the moving load analysis to the nominal and factored capacities for moment and shear. The entire demand history shown in Fig. 9 is not required to rate the bridge girder for individual force effects according to Eq. (4); however, it is required when rating for the interaction of moment and shear (Higgins et al. 2005).

## Conclusions

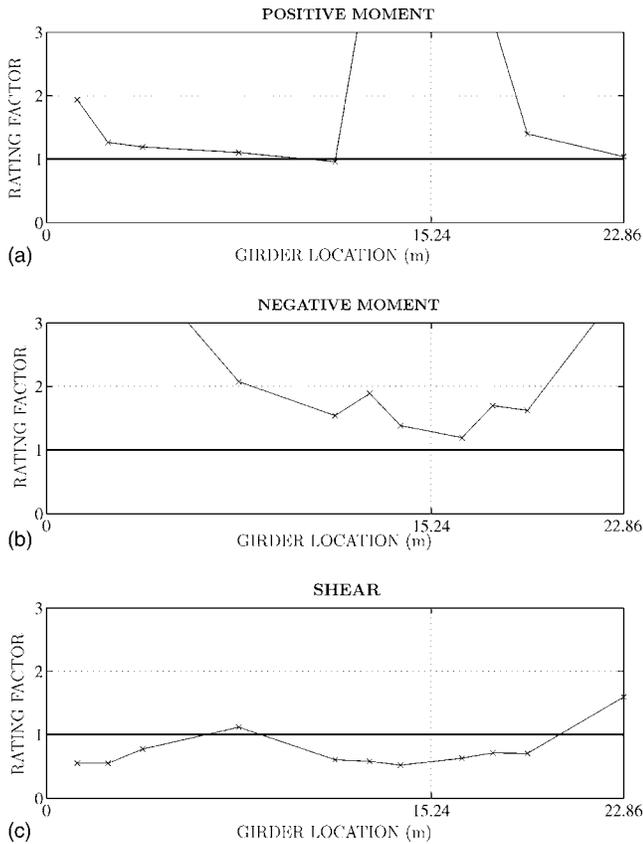
A software application for bridge rating was developed in which the Tcl scripting language glues the finite-element analysis modules of OpenSees together to create customized procedures for live load analysis, bridge capacity, and rating factor calculations. The application was demonstrated for rating a conventionally reinforced concrete bridge girder for moment and shear effects; however, it is clear that the application is extensible to other structural components and bridge types. Further development will make the rating application a powerful tool to help prioritize highway bridges for repair, rehabilitation, and/or replacement.

As noted in the foregoing discussion, the following benefits are realized by using Tcl to build the bridge rating application:

1. All steps in the application are transparent to a user, who is thus not bound to specific capacity models and rating formats and is free to experiment with existing bridge rating methodologies, e.g., for site-specific load factors;



**Fig. 7.** Load configuration for Oregon single trip permit vehicle, OR-STP-5B (adapted from *Oregon Department of Transportation Load Rating Manuals*)



**Fig. 8.** LRF rating factors for OR-STP-5B load configuration on interior girder of McKenzie River Bridge considering individual force effects: (a) positive moment; (b) negative moment; and (c) shear

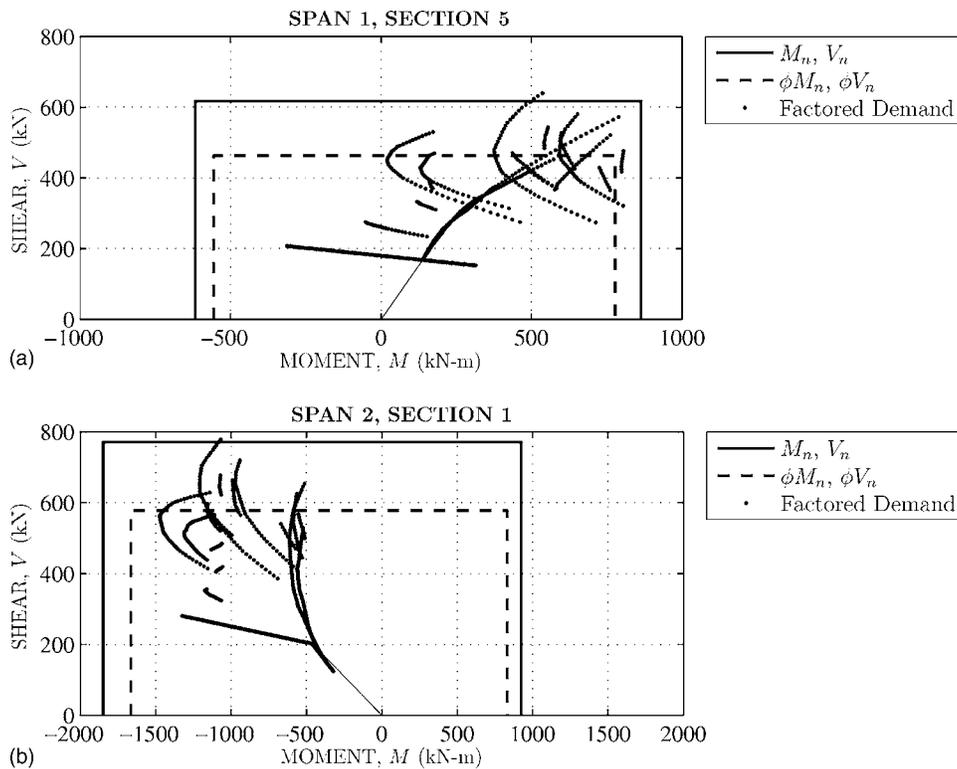
2. Procedures that implement bridge capacity models and rating factor calculations are portable to multiple computing architectures; and
3. In an interpreted environment such as Tcl, the compile phase of the edit-compile-debug cycle is removed, potentially decreasing development times for emergent bridge rating methodologies.

In addition to the benefits offered by Tcl, the following extensions of the bridge rating application are foreseen:

1. The simulation of bridge overload conditions using the non-linear element and constitutive models of OpenSees;
2. Component and system reliability analysis for multihazard effects of vehicle, seismic, wave, and blast loads;
3. Sensitivity-based applications including structural reliability, optimization, and system identification are possible using the framework for finite-element reliability and sensitivity analysis in OpenSees; and
4. Analyzing and rating bridges in a transportation network while taking advantage of the parallel and distributed computing capabilities of OpenSees.

Given the combined capabilities of Tcl and OpenSees, the bridge rating application presented in this paper forms the basis for a comprehensive bridge management system that incorporates the latest advances in simulation models and information technology. For example, the underlying architecture of Tcl allows the application to communicate efficiently with network servers in order to use weigh-in-motion data for structural health monitoring.

The particular choices of OpenSees for finite-element analysis and Tcl as a scripting language do not detract from the underlying concept that readily available software can be easily combined to solve practical engineering problems. While several commercially available finite-element software packages are scriptable



**Fig. 9.** Demand history and capacity envelopes at two critical locations for OR-STP-5B load configuration applied to interior girder of McKenzie River Bridge: (a) 11.6 m along span 1; (b) 1.22 m along span 2

and could have been used to build bridge rating applications, OpenSees was chosen since it is available over the Internet, (<http://opensees.berkeley.edu>) for download by any interested party. All Tcl scripts presented in this paper are also available for download at the corresponding writer's web page, (<http://web.engr.oregonstate.edu/~mhscott>).

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