Response Sensitivity for Nonlinear Beam–Column Elements

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Abstract: Response sensitivity is needed for simulation applications such as optimization, reliability, and system identification. The exact response sensitivity of material nonlinear beam–column elements is derived for the displacement- and force-based formulations. For displacement-based beam–column elements the response sensitivity is straightforward to compute because the displacement field is specified along the element. A new approach is presented for computing the response sensitivity of force-based beam–column elements, in which the displacement field is not specified. In this approach, the response sensitivity depends on the derivative of unbalanced section forces because the element displacement field changes with the element state. Example nonlinear static analyses of steel and reinforced concrete structural systems verify the exact response sensitivity for force-based elements using the new approach.

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Introduction

The computational simulation of structural systems has an integral role in earthquake engineering analysis and design. The ability to model accurately the response of a structural system is crucial, particularly with the advent of performance-based earthquake engineering methodologies. The simulated response of a structure depends on modeling assumptions, a majority of which are based on simplified approximations. In addition, the design of a new structural system is a function of design parameters having uncertain properties. The computation of response sensitivity provides guidance to engineers as to which parameters control the response of a structural system. As a consequence, the sensitivity of the system response is just as important as the response itself. Structural reliability, optimization, and system identification applications all require accurate and efficient response sensitivity computations for the convergence of iterative search algorithms to an optimal solution point (Liu and Der Kiureghian 1991).

The sensitivity of a structural response quantity due to a change in a parameter can be computed in one of two ways. The first approach is the finite difference method (FDM), in which the simulation is repeated with a perturbed value of the parameter. This approach is time consuming because the analysis must be repeated for each parameter that defines the model, and it is prone to numerical round-off error for small parameter perturbations. The second method is the direct differentiation method (DDM), in which the governing equations of structural equilibrium, compatibility, and constitution are differentiated exactly (Kleiber et al. 1997). The DDM gives the response sensitivity as the analysis proceeds, rather than by reanalysis with perturbed parameters, and it can be computed very efficiently.

In the nonlinear analysis of structural systems, there are two types of formulations for beam–column elements: displacement-based and force-based. Displacement-based elements follow the standard finite element procedure of specifying an approximate displacement field along the element (Zienkiewicz and Taylor 2000). In contrast, force-based elements interpolate internal forces, which is exact even in the nonlinear range of material behavior (Spacone et al. 1996). Neuenhofer and Filippou (1997) have stated the advantages of force-based elements over displacement-based elements, the most notable being the ability to use one element to represent the material nonlinear behavior of a beam–column member, compared with several displacement-based elements for a single member. The formulation of response sensitivity for displacement-based elements is straightforward because the element displacement field is specified. The derivation of response sensitivity for force-based elements, however, is more difficult because the displacement field depends on the section constitutive response, which must be determined such that equilibrium is maintained with the element forces.

The objective of this paper is to present a uniform approach to the computation of response sensitivity for beam–column elements, highlighting the similarities and differences between the displacement- and force-based formulations. The methodology for computing the response sensitivity at the structural level is formulated first. Then the response sensitivity of displacement-based elements is presented, followed by a new derivation of the response sensitivity for force-based elements. Attention then turns to the computation of response sensitivity at the section constitutive level. The paper concludes with example applications showing the validity of the force-based element response sensitivity.

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Global Formulation for Sensitivity

The global equilibrium equations for a structural system whose resisting forces come from inelastic rate-independent material models have the general form

$$\mathbf{P}_r(\mathbf{U}(\mathbf{\Theta},t),\mathbf{\Theta})=\mathbf{P}_f(\mathbf{\Theta},t) \tag{1}$$

where $\mathbf{P}_f$ is a vector of the external forces applied to the structure; and $\mathbf{\Theta}$ is a vector of parameters for the structural model. The external forces are a function of pseudo-time, t, as is the nodal displacement vector $\mathbf{U}$, which is obtained by standard nonlinear solution procedures such as the Newton–Raphson method. The resisting force vector $\mathbf{P}_r$ is assembled from element forces, and it depends on the parameters explicitly, as well as implicitly via the nodal displacements. The focus of this paper is the derivation of response sensitivity for the element resisting forces. As a result, Eq. (1) is limited to static equilibrium effects because the inclusion of inertial and damping forces for nonlinear dynamic analysis is accomplished at the global level following well established procedures, and is independent of the element formulation.

Applying the chain rule to Eq. (1), the derivative of the equilibrium equations with respect to a parameter θ, which belongs to the vector $\mathbf{\Theta}$, is

$$\frac{\partial \mathbf{P}_r}{\partial \mathbf{U}}\frac{\partial \mathbf{U}}{\partial \theta}+\left.\frac{\partial \mathbf{P}_r}{\partial \theta}\right|_{\mathbf{U}}=\frac{\partial \mathbf{P}_f}{\partial \theta} \tag{2}$$

where $\partial \mathbf{P}_r/\partial\theta|_{\mathbf{U}}$ is termed the conditional derivative of the resisting force vector because it gives the derivative of the resisting forces with respect to θ while the displacements are held fixed. Physically, this derivative represents the change in resisting forces required to keep the structure fixed at the current state while the parameter changes. Using the definition of the tangent stiffness matrix, $\mathbf{K}_T=\partial \mathbf{P}_r/\partial \mathbf{U}$, Eq. (2) gives a linear system of equations for the nodal response sensitivity $\partial \mathbf{U}/\partial\theta$

$$\mathbf{K}_T\frac{\partial \mathbf{U}}{\partial \theta}=\frac{\partial \mathbf{P}_f}{\partial \theta}-\left.\frac{\partial \mathbf{P}_r}{\partial \theta}\right|_{\mathbf{U}} \tag{3}$$

The conditional derivative of the resisting force vector is assembled from element contributions in the same manner as the resisting force vector. Response sensitivity analysis at the global level requires assembly of the right-hand side and solution of the factorized linear system of equations [Eq. (3)] for each parameter in the vector $\mathbf{\Theta}$. The derivative of the external force vector, $\partial \mathbf{P}_f/\partial\theta$, is nonzero only for parameters that describe the load applied to the structure. With the formulation at the global level, the element contribution to the response sensitivity must be determined.

Element Formulation for Sensitivity

Beam–column elements are conveniently formulated in a basic system, free of rigid body displacement modes. For this discussion, element deformations are assumed small. In the two-dimensional simply supported basic system, the element deformation vector, $\mathbf{v}=\mathbf{v}(\theta)$, consists of three components: one axial deformation and one rotation at each node, as shown in Fig. 1. Three-dimensional elements have six deformations. The corresponding basic forces, $\mathbf{q}=\mathbf{q}(\mathbf{v}(\theta),\theta)$, are a function of the element deformations, as well as the parameter θ. At every cross section along the element, there are section deformations, $\mathbf{e}=\mathbf{e}(\theta)$, and the corresponding section forces, $\mathbf{s}=\mathbf{s}(\mathbf{e}(\theta),\theta)$. For both the displacement- and force-based element formulations, the element response is obtained by integration of the section response, as prescribed by the equations of element equilibrium and compatibility.

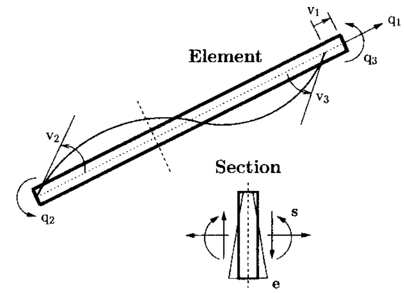

Fig. 1. Simply supported basic system for two-dimensional beam–column elements

To compute the element response sensitivity, it is necessary to differentiate the basic and section forces with respect to the parameter θ. By application of the chain rule, in a manner similar to the global resisting forces, the derivative of the basic forces is

$$\frac{\partial \mathbf{q}}{\partial \theta}=\mathbf{k}\frac{\partial \mathbf{v}}{\partial \theta}+\left.\frac{\partial \mathbf{q}}{\partial \theta}\right|_{\mathbf{v}} \tag{4}$$

where $\mathbf{k}=\partial\mathbf{q}/\partial\mathbf{v}$ is the element stiffness matrix. The derivative of the section forces is also obtained by the chain rule

$$\frac{\partial \mathbf{s}}{\partial \theta}=\mathbf{k}_s\frac{\partial \mathbf{e}}{\partial \theta}+\left.\frac{\partial \mathbf{s}}{\partial \theta}\right|_{\mathbf{e}} \tag{5}$$

where the section stiffness matrix is $\mathbf{k}_s=\partial\mathbf{s}/\partial\mathbf{e}$.

The geometric transformations of basic forces $\mathbf{q}$ to global resisting forces $\mathbf{P}_r$ and global displacements $\mathbf{U}$ to element deformations $\mathbf{v}$ are linear for small deformations and displacements. These transformations are carried out by well documented structural analysis procedures, and the response derivatives transform between the global and basic systems in the same manner as the response itself.

The computation of response sensitivity for path-dependent problems is a two-phase process (Zhang and Der Kiureghian 1993). Phase one begins with the assembly of $\partial \mathbf{P}_r/\partial\theta|_{\mathbf{U}}$ from the conditional derivative of the basic forces $\partial\mathbf{q}/\partial\theta|_{\mathbf{v}}$ for each element. Solution for the nodal response sensitivity $\partial\mathbf{U}/\partial\theta$ by Eq. (3) concludes phase one. For path-dependent problems, the response sensitivity is also path-dependent. To track the path-dependent response sensitivity, sensitivity history variables are required. The computation of the derivative of section deformations $\partial\mathbf{e}/\partial\theta$ from the nodal response sensitivity permits the update of these sensitivity history variables during phase two. Which sensitivity history variables must be stored and how they are updated depends on the section constitutive model. This process is outlined by Zhang and Der Kiureghian (1993) for the J_2 plasticity model (Simo and Hughes 1998) and in the Appendix of this paper for a simplified uniaxial concrete model.

Due to the governing equations of element equilibrium and compatibility, the computation of the conditional derivative of basic forces and the derivative of section deformations is different for displacement- and force-based elements. The statements of equilibrium and compatibility, and the computational steps for the response sensitivity, are presented in the following two sections for each element formulation.

Displacement-Based Elements

For displacement-based beam–column elements (Zienkiewicz and Taylor 2000), compatibility along the element is stated as

$$\mathbf{e}=\mathbf{a}_e\mathbf{v} \tag{6}$$

where the matrix $\mathbf{a}_e=\mathbf{a}_e(x)$ contains interpolation functions relating section deformations to element deformations. The principle of virtual displacements leads to a weak form of equilibrium between basic forces and section forces

$$\mathbf{q}=\int_0^L \mathbf{a}_e^T\mathbf{s}dx \tag{7}$$

The element stiffness matrix is obtained from linearization of Eq. (7) with respect to the element deformations

$$\mathbf{k}=\frac{\partial \mathbf{q}}{\partial \mathbf{v}}=\int_0^L \mathbf{a}_e^T\mathbf{k}_s\mathbf{a}_e dx \tag{8}$$

Typically, the assumed displacement fields along the element are linear for the axial component and cubic Hermitian for the transverse component. These displacement fields admit constant axial deformation and linear curvature along the element in Eq. (6), according to the Bernoulli beam theory. Due to this approximation of deformations, which is exact only for linear elastic, prismatic elements, it is necessary to use several displacement-based elements (h refinement) to represent the material nonlinear behavior of a structural member. It is possible to use higher order interpolation functions (p refinement), but the deformations along the element remain constrained to an approximate and generally inaccurate solution for material nonlinearity.

The response sensitivity for displacement-based elements is well known (Zhang and Der Kiureghian 1993), but it is derived in this paper by an approach that lends insight into the derivation of response sensitivity for force-based elements. To determine the response sensitivity, the equilibrium relationship, Eq. (7), is differentiated with respect to θ

$$\frac{\partial \mathbf{q}}{\partial \theta}=\int_0^L \mathbf{a}_e^T\frac{\partial \mathbf{s}}{\partial \theta}dx \tag{9}$$

After insertion of the basic and section force derivatives, from Eqs. (4) and (5), Eq. (9) expands to

$$\mathbf{k}\frac{\partial \mathbf{v}}{\partial \theta}+\left.\frac{\partial \mathbf{q}}{\partial \theta}\right|_{\mathbf{v}}=\int_0^L \mathbf{a}_e^T\left(\mathbf{k}_s\frac{\partial \mathbf{e}}{\partial \theta}+\left.\frac{\partial \mathbf{s}}{\partial \theta}\right|_{\mathbf{e}}\right)dx \tag{10}$$

The solution for the conditional derivative of basic forces gives

$$\left.\frac{\partial \mathbf{q}}{\partial \theta}\right|_{\mathbf{v}}=\int_0^L \mathbf{a}_e^T\left.\frac{\partial \mathbf{s}}{\partial \theta}\right|_{\mathbf{e}}dx+\int_0^L \mathbf{a}_e^T\mathbf{k}_s\frac{\partial \mathbf{e}}{\partial \theta}dx-\mathbf{k}\frac{\partial \mathbf{v}}{\partial \theta} \tag{11}$$

The derivative of the element compatibility relationship, Eq. (6)

$$\frac{\partial \mathbf{e}}{\partial \theta}=\mathbf{a}_e\frac{\partial \mathbf{v}}{\partial \theta} \tag{12}$$

is combined with Eq. (11) to give the following expression:

$$\left.\frac{\partial \mathbf{q}}{\partial \theta}\right|_{\mathbf{v}}=\int_0^L \mathbf{a}_e^T\left.\frac{\partial \mathbf{s}}{\partial \theta}\right|_{\mathbf{e}}dx+\left(\int_0^L \mathbf{a}_e^T\mathbf{k}_s\mathbf{a}_e dx\right)\frac{\partial \mathbf{v}}{\partial \theta}-\mathbf{k}\frac{\partial \mathbf{v}}{\partial \theta} \tag{13}$$

From the definition of the element stiffness matrix, Eq. (8), the last two terms on the right-hand side of Eq. (13) are equal, and the conditional derivative of the basic forces for the displacement-based formulation reduces to

$$\left.\frac{\partial \mathbf{q}}{\partial \theta}\right|_{\mathbf{v}}=\int_0^L \mathbf{a}_e^T\left.\frac{\partial \mathbf{s}}{\partial \theta}\right|_{\mathbf{e}}dx \tag{14}$$

In the implementation of displacement-based beam–column elements, the basic forces and their conditional derivative, from Eqs. (7) and (14), respectively, are evaluated by the Gauss–Legendre numerical integration rule. For the assumption of a cubic Hermitian transverse displacement field along the element, two Gauss–Legendre integration points is sufficient.

The result in Eq. (14) could have been obtained directly from Eq. (9) because the condition of fixed nodal displacements leads to fixed element and section deformations in the displacement-based element formulation. However, differentiation of the element equilibrium and compatibility relationships and subsequent combination of these derivatives is necessary in the derivation of response sensitivity for force-based elements.

Force-Based Elements

In the force-based element formulation (Spacone et al. 1996), the equilibrium relationship is stated in strong form as

$$\mathbf{s}=\mathbf{b}\mathbf{q} \tag{15}$$

where the matrix $\mathbf{b}=\mathbf{b}(x)$ contains equilibrium interpolation functions that express section forces in terms of basic forces. In the absence of element loads, the axial force is constant, while the bending moment varies linearly along the element.

From the principle of virtual forces, the compatibility relationship between section deformations and element deformations is

$$\mathbf{v}=\int_0^L \mathbf{b}^T\mathbf{e}dx \tag{16}$$

Linearization of Eq. (16) with respect to basic forces gives the element flexibility matrix

$$\mathbf{f}=\frac{\partial \mathbf{v}}{\partial \mathbf{q}}=\int_0^L \mathbf{b}^T\mathbf{f}_s\mathbf{b}dx \tag{17}$$

where $\mathbf{f}_s=\mathbf{k}_s^{-1}$ is the section flexibility matrix. Inversion of the element flexibility matrix gives the element stiffness matrix, $\mathbf{k}=\mathbf{f}^{-1}$.

The derivation of response sensitivity for force-based elements is not as straightforward as that for displacement-based elements. The difficulty arises from the structure of the element equilibrium and compatibility relationships in the force-based formulation. To demonstrate this difficulty, differentiation of the equilibrium relationship, Eq. (15), with respect to θ gives

$$\frac{\partial \mathbf{s}}{\partial \theta}=\mathbf{b}\frac{\partial \mathbf{q}}{\partial \theta} \tag{18}$$

After substitution of the derivatives of basic and section forces from Eqs. (4) and (5), Eq. (18) expands to

$$\mathbf{k}_s\frac{\partial \mathbf{e}}{\partial \theta}+\left.\frac{\partial \mathbf{s}}{\partial \theta}\right|_{\mathbf{e}}=\mathbf{b}\left(\mathbf{k}\frac{\partial \mathbf{v}}{\partial \theta}+\left.\frac{\partial \mathbf{q}}{\partial \theta}\right|_{\mathbf{v}}\right) \tag{19}$$

However, the conditional derivative of basic forces cannot be determined from Eq. (19) because the interpolation matrix $\mathbf{b}$ is not invertible. Adding to the difficulty, differentiation of the element compatibility relationship, Eq. (16), gives

$$\frac{\partial \mathbf{v}}{\partial \theta}=\int_0^L \mathbf{b}^T\frac{\partial \mathbf{e}}{\partial \theta}dx \tag{20}$$

Table 1. Steps for Computation of Beam–Column Element Response Sensitivity

Phase	Step	Force-based	Displacement-based
I	Section flexibility	$\mathbf{f}_S=\mathbf{k}_s^{-1}$	
	Element flexibility	$\mathbf{f}=\int_0^L \mathbf{b}^T\mathbf{f}_s\mathbf{b}dx$	
	Element stiffness	$\mathbf{k}=\mathbf{f}^{-1}$	
	Section force conditional derivative	$\left.\frac{\partial \mathbf{s}}{\partial\theta}\right\|_{\mathbf{e}}$	$\left.\frac{\partial \mathbf{s}}{\partial\theta}\right\|_{\mathbf{e}}$
	Basic force conditional derivative	$\left.\frac{\partial \mathbf{q}}{\partial\theta}\right\|_{\mathbf{v}}=\mathbf{k}\int_0^L \mathbf{b}^T\mathbf{f}_s\left.\frac{\partial \mathbf{s}}{\partial\theta}\right\|_{\mathbf{e}}dx$	$\left.\frac{\partial \mathbf{q}}{\partial\theta}\right\|_{\mathbf{v}}=\int_0^L \mathbf{a}_e^T\left.\frac{\partial \mathbf{s}}{\partial\theta}\right\|_{\mathbf{e}}dx$
	Assembly and solution for nodal response sensitivity $\frac{\mathbf{K}_T\partial \mathbf{U}}{\partial\theta}=\frac{\partial \mathbf{P}_f}{\partial\theta}-\left.\frac{\partial \mathbf{P}_r}{\partial\theta}\right\|_{\mathbf{U}}$		
II	Element deformation derivative	$\frac{\partial \mathbf{v}}{\partial\theta}$	$\frac{\partial \mathbf{v}}{\partial\theta}$
	Section deformation derivative	$\frac{\partial \mathbf{e}}{\partial\theta}=\mathbf{f}_s\mathbf{bk}\frac{\partial \mathbf{v}}{\partial\theta}+\mathbf{f}_s\left(\mathbf{b}\left.\frac{\partial \mathbf{q}}{\partial\theta}\right\|_{\mathbf{v}}-\left.\frac{\partial \mathbf{s}}{\partial\theta}\right\|_{\mathbf{e}}\right)$	$\frac{\partial \mathbf{e}}{\partial\theta}=\mathbf{a}_e\frac{\partial \mathbf{v}}{\partial\theta}$

in which the derivative of the section deformations appears in the integrand on the right-hand side of Eq. (20). From the form of Eqs. (19) and (20), it is apparent that further manipulation is required to derive the conditional derivative of the basic forces. The following derivation establishes the response sensitivity for force-based elements.

As a first step, the derivative of the section deformations can be obtained from Eq. (19) by solving for $\partial\mathbf{e}/\partial\theta$

$$\frac{\partial \mathbf{e}}{\partial\theta}=\mathbf{f}_s\mathbf{bk}\frac{\partial \mathbf{v}}{\partial\theta}+\mathbf{f}_s\left(\mathbf{b}\left.\frac{\partial \mathbf{q}}{\partial\theta}\right|_{\mathbf{v}}-\left.\frac{\partial \mathbf{s}}{\partial\theta}\right|_{\mathbf{e}}\right) \tag{21}$$

The quantity in parentheses on the right-hand side of Eq. (21) represents an unbalanced section force derivative because it is the difference between the interpolation of the conditional derivative of the basic forces and the conditional derivative of the section forces. The conditional derivative of the basic forces must be computed, and the key step is the substitution of Eq. (21) into Eq. (20)

$$\frac{\partial \mathbf{v}}{\partial\theta}=\left(\int_0^L \mathbf{b}^T\mathbf{f}_s\mathbf{b}dx\right)\mathbf{k}\frac{\partial \mathbf{v}}{\partial\theta}+\int_0^L \mathbf{b}^T\mathbf{f}_s\left(\mathbf{b}\left.\frac{\partial \mathbf{q}}{\partial\theta}\right|_{\mathbf{v}}-\left.\frac{\partial \mathbf{s}}{\partial\theta}\right|_{\mathbf{e}}\right)dx \tag{22}$$

From the definition of the element flexibility matrix in Eq. (17) and the identity $\mathbf{fk}=\mathbf{I}$, the term on the left-hand side and the first term on the right-hand side of Eq. (22) are equal. Eq. (22) then reduces to

$$\mathbf{0}=\int_0^L \mathbf{b}^T\mathbf{f}_s\left(\mathbf{b}\left.\frac{\partial \mathbf{q}}{\partial\theta}\right|_{\mathbf{v}}-\left.\frac{\partial \mathbf{s}}{\partial\theta}\right|_{\mathbf{e}}\right)dx \tag{23}$$

Eq. (23) is important because it requires the unbalanced section force derivatives to be zero in an average sense along the element. After inversion of the element flexibility matrix in Eq. (23), the final expression for the conditional derivative of the basic forces in the force-based formulation is

$$\left.\frac{\partial \mathbf{q}}{\partial\theta}\right|_{\mathbf{v}}=\mathbf{k}\int_0^L \mathbf{b}^T\mathbf{f}_s\left.\frac{\partial \mathbf{s}}{\partial\theta}\right|_{\mathbf{e}}dx \tag{24}$$

It is worth noting the similarity between Eqs. (14) and (24) for the two beam–column formulations since it can be shown that the interpolation matrix relating the increment in section deformations to an increment in element deformations is $\mathbf{a}_e=\mathbf{f}_s\mathbf{bk}$ for force-based elements. The interpolation of section deformations from element deformations depends on the current element state because the element stiffness changes due to nonlinearity in the section constitutive response. As a result, the unbalanced section force derivative must be included in the derivative of the section deformations, as seen in Eq. (21). For displacement-based elements, the derivative of the section deformations is computed directly from the derivative of the element deformations in Eq. (12) according to the specified element displacement field.

In the numerical implementation of force-based beam–column elements, the compatibility relationship, Eq. (16), must be solved by an iterative method, e.g., Newton–Raphson, or by a noniterative approach (Neuenhofer and Filippou 1997). The intermediate steps in finding element compatibility do not affect the response sensitivity for these elements because the sensitivity is only computed at a converged state in which compatibility is satisfied. To assure the computation of the exact response sensitivity for force-based elements, the consistent section flexibility and element stiffness matrices from the compatible state must be used in Eqs. (21) and (24). Furthermore, the Gauss–Lobatto numerical integration rule is applied to Eqs. (16) and (24) because it places integration points at the element ends, where the bending moments are largest in the absence of element loads. Typically, three to five Gauss–Lobatto integration points along the element accurately represent the material nonlinear behavior.

The two-phase process for the computation of response sensitivity for beam–column elements is summarized in Table 1. The conditional derivative of the basic forces is computed by either Eq. (14) or (24). Then, after assembly and solution for the nodal response sensitivity in Eq. (3), the derivative of the section deformations is determined by Eq. (12) or (21). The computational steps outlined in Table 1 have been implemented in the *OpenSees* finite element analysis framework (McKenna et al. 2000).

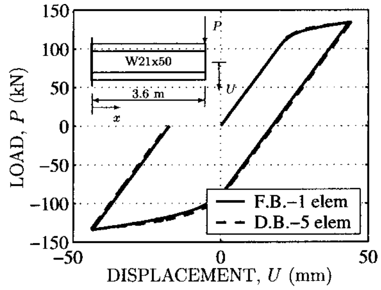

Fig. 2. Load–displacement response of steel cantilever beam for force- and displacement-based beam–column elements

Section Formulation for Sensitivity

Having formulated the response sensitivity at the global and element levels, attention turns to the formulation at the section level. There are two methods to specify the section constitutive response. The first method is the specification of the section forces as a function of the section deformations by a resultant plasticity model. Direct differentiation of the section response gives the conditional derivative of the section forces, $\partial \mathbf{s}/\partial\theta|_{\mathbf{e}}$.

The second method is integration of the material stress response σ over the section area

$$\mathbf{s}=\int_A \mathbf{a}_s^T \sigma\, dA \tag{25}$$

The compatibility matrix $\mathbf{a}_s$ gives the material strain ε as a function of the section deformations

$$\varepsilon=\mathbf{a}_s\mathbf{e} \tag{26}$$

By a derivation identical to that for displacement-based elements, the conditional derivative of the section forces for this method is

$$\left.\frac{\partial \mathbf{s}}{\partial\theta}\right|_{\mathbf{e}}=\int_A \mathbf{a}_s^T \left.\frac{\partial\sigma}{\partial\theta}\right|_{\varepsilon} dA \tag{27}$$

Differentiation of the statement of section compatibility, Eq. (26), gives the relationship between the derivatives of material strain and section deformations

$$\frac{\partial\varepsilon}{\partial\theta}=\mathbf{a}_s\frac{\partial \mathbf{e}}{\partial\theta} \tag{28}$$

This relationship is necessary to account for path-dependent behavior in the computation of response sensitivity at the material level.

Examples

To show the validity of the new DDM for the force-based element formulation, the response sensitivity is compared with that obtained from the FDM for small parameter perturbations, $\Delta\theta=\varepsilon\theta$. For decreasing parameter perturbations, the sensitivity obtained by the FDM should converge to the DDM sensitivity

$$\lim_{\Delta\theta\to 0}\frac{\mathbf{U}(\theta+\Delta\theta)-\mathbf{U}(\theta)}{\Delta\theta}=\frac{\partial\mathbf{U}}{\partial\theta} \tag{29}$$

The convergence of the FDM to the DDM indicated in Eq. (29) is demonstrated in the following nonlinear static analyses of steel and reinforced concrete structural systems.

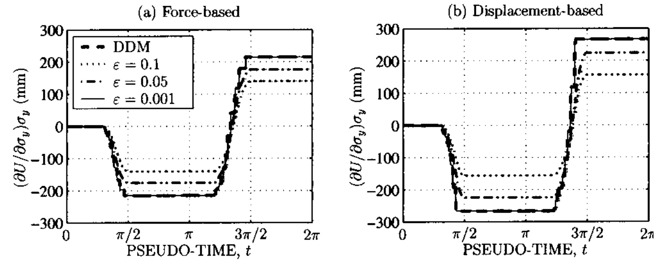

Fig. 3. Steel cantilever beam response sensitivity through one sinusoidal load cycle with respect to yield stress computed by direct differentiation and finite difference methods: (a) force based, one element and (b) displacement based, five elements

Steel Cantilever

In the first example, a cantilever steel beam is used to compare the displacement- and force-based formulations. The response of the $W21\times 50$ section is integrated by the midpoint rule with 24 layers from the stress–strain behavior represented by a uniaxial version of the J_2 plasticity model. The elastic modulus is $E=2.0\times 10^5$ MPa, the yield stress is $\sigma_y=250$ MPa, and 2% kinematic strain hardening is assumed. The beam is loaded at its free end through one sinusoidal cycle of peak intensity $P=134$ kN.

The load–displacement response of the steel cantilever is shown in Fig. 2. A mesh of five displacement-based elements captures the material nonlinear behavior of the beam where only one force-based element is required. The response sensitivity is computed with respect to the steel yield stress for both element formulations, and is shown in Fig. 3. The results obtained by the FDM converge to those obtained by the DDM for both the displacement- and force-based element formulations. Discrete jumps are seen in the sensitivity as the steel material switches from elastic to plastic states (Conte et al. 1999).

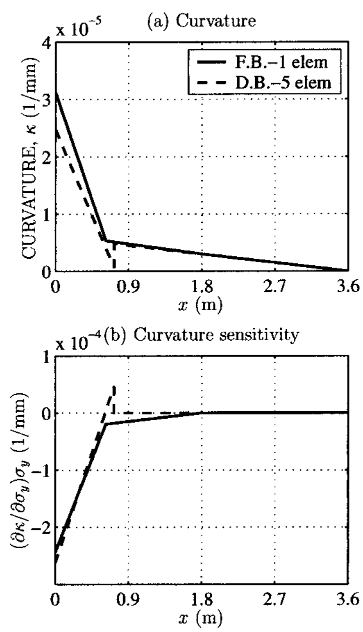

Fig. 4. Steel cantilever beam local response at the peak load for force- and displacement-based element formulations: (a) curvature distribution and (b) sensitivity of curvature distribution with respect to yield stress, computed by direct differentiation method

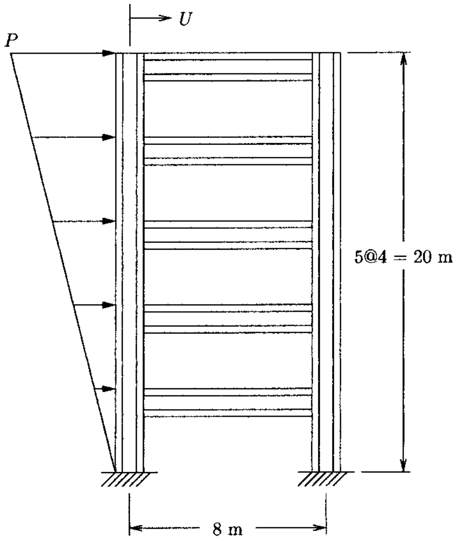

Fig. 5. Steel moment-resisting frame model for sensitivity example

An application of response sensitivity to structural design is to determine the change in displacement resulting from a change in the selected parameter. Plotted in Fig. 3 is the quantity $(\partial U/\partial \sigma_y)\sigma_y$, the sensitivity of the tip displacement with respect to yield stress, scaled by the yield stress. Multiplication of this quantity by a percent change in yield stress gives the resulting change in tip displacement. For example, at the peak load of 134 kN, the tip displacement is 43.4 mm for the force formulation, and the scaled response sensitivity computed by the DDM is -214.8 mm. A 5% increase in yield stress gives a 10.7 mm (25%) reduction in the tip displacement.

Although the global response of the steel beam agrees very well for the displacement- and force-based element formulations, the response sensitivity is quite different, as seen in Fig. 3. To assess this discrepancy in response sensitivity, the local response of the beam is examined. There is a noticeable difference in the distribution of curvature in the plastic hinge zone at the peak load of 134 kN, as shown in Fig. 4(a). The scaled response sensitivity of the curvature distribution at the peak load is shown in Fig. 4(b). At the peak load, the maximum curvature predicted by the force formulation is 3.12×10^{-5}/mm, and the associated scaled response sensitivity is -2.44×10^{-4}/mm. A reduction of 1.22 $\times10^{-5}$/mm (39%) in maximum curvature is estimated from a 5% increase in the steel yield stress.

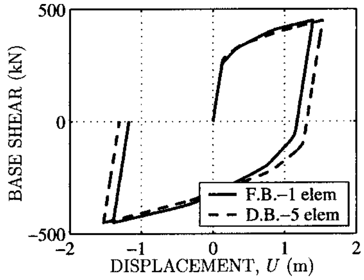

Fig. 6. Load–displacement response of steel moment-resisting frame for force- and displacement-based beam–column elements

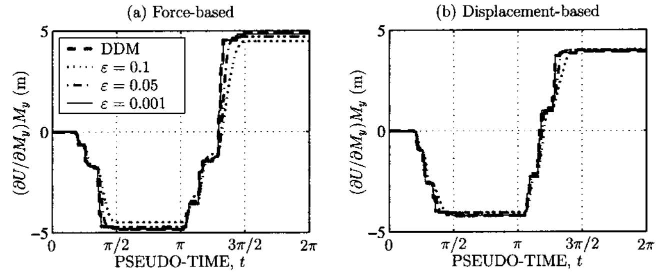

Fig. 7. Steel frame response sensitivity through one sinusoidal load cycle with respect to yield moment computed by direct differentiation and finite difference methods: (a) force based, one element per member and (b) displacement based, five elements per member

Steel Moment-Resisting Frame

The second example is a five-story, one-bay steel moment-resisting frame, as shown in Fig. 5. All members are $W21\times50$ with $E=2.0\times10^5$ MPa and $\sigma_y=250$ MPa. The flexural behavior is represented by a bilinear moment–curvature relationship with 2% kinematic hardening. The axial behavior of the members is assumed to be linear elastic and uncoupled from the flexural behavior. The frame is loaded laterally through one sinusoidal cycle by an inverted triangular distribution. The maximum force applied at the roof level is $P=150$ kN, which gives a peak base shear of 450 kN for the given lateral load distribution.

The response is computed with one element per member for the force-based formulation (five Gauss–Lobatto integration points), and five elements per member for the displacement-based formulation. The base shear is plotted against the roof displacement in Fig. 6 for both element formulations. The sensitivity of the roof displacement with respect to the yield moment, M_y, of the $W21\times50$ section, is shown in Fig. 7. The sensitivity obtained by the FDM converges to the DDM sensitivity and discrete jumps appear in the sensitivity as plastic hinges form throughout the structure. This example shows the validity of the response sensitivity computation for elements in which a resultant plasticity model represents the section behavior, rather than the integration of material stress, as in the first example.

Reinforced Concrete Cantilever Beam

The final example is of a reinforced concrete cantilever beam. The section is rectangular, with two layers of five No. 9 steel bars

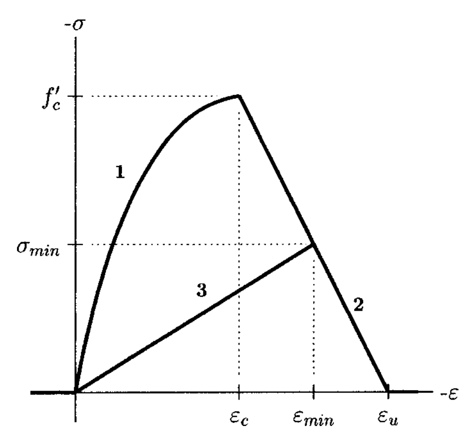

Fig. 8. Stress–strain relationship for uniaxial concrete model

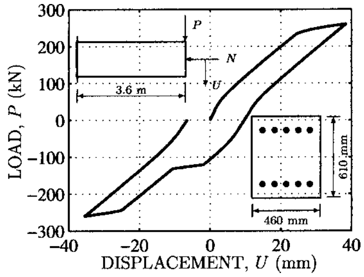

Fig. 9. Load–displacement response of reinforced concrete cantilever beam for one force-based element

and 50 mm cover. The concrete is represented by the phenomenological stress–strain relationship shown in Fig. 8, in which monotonic behavior in compression is represented by a parabolic function up to the peak compressive strength of f_c' at a strain of ε_c, followed by a linear descending branch to a crushing strain of ε_u, where the concrete loses all strength. For simplicity, the concrete model unloads and reloads along the same branch, which passes through the origin, and the tensile strength is zero. The derivatives of the stress–strain response for this concrete material model are presented in the Appendix.

For this example, the concrete has a compressive strength of $f_c' = -28.0$ MPa, crushing strain $\varepsilon_c = -0.002$, and ultimate strain $\varepsilon_u = -0.006$. The steel is bilinear with 2% kinematic hardening, elastic modulus $E = 2.0\times10^5$ MPa, and yield stress $\sigma_y = 420$ MPa. A constant axial load equal to 10% of the gross section capacity and a transverse sinusoidal load of peak intensity $P = 260$ kN are applied to the beam. The integration of the steel and concrete material stress response over the section area accounts for axial–moment interaction, and the midpoint integration rule is used with 20 concrete layers.

The load–displacement response for the beam is shown in Fig. 9. The response sensitivity is computed with respect to both the steel yield stress and the concrete compressive strength for a single force-based element with five Gauss–Lobatto integration points. As shown in Fig. 10, the beam displacement is much more sensitive to the steel yield stress than to the concrete compressive strength.

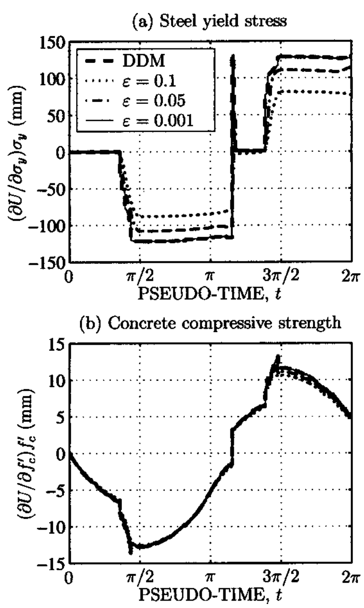

Fig. 10. Reinforced concrete cantilever beam response sensitivity through one sinusoidal load cycle with respect to: (a) Steel yield stress and (b) concrete compressive strength; computed by direct differentiation and finite difference methods for one force-based element

Conclusions

The exact response sensitivity of force-based beam–column elements has been developed from a consistent definition of the derivatives of the element equilibrium, compatibility, and section force–deformation relationships. Direct differentiation of the governing statements of element equilibrium and compatibility provides a uniform approach to computing the response sensitivity for both displacement- and force-based elements. This approach overcomes the difficulty that arises in the derivation of response sensitivity for force-based elements because the displacement field along the element is not specified, but changes according to the element state.

The response sensitivity computed by the finite difference method converges to the exact sensitivity computed by the direct differentiation method, as demonstrated by the nonlinear analysis of steel and reinforced concrete structures. The ability to compute the response sensitivity accurately and efficiently for force-based elements broadens the application of these elements in the fields of structural analysis, design, reliability, and optimization.

Appendix. Uniaxial Concrete Model

The response derivatives for the uniaxial concrete material model shown in Fig. 8 are presented in this Appendix. There are two history variables to track path-dependent behavior: $\varepsilon_{\min}$, the largest compressive strain, and $\sigma_{\min}$, the stress on the backbone corresponding to $\varepsilon_{\min}$. The only parameter considered for differentiation in this example is the compressive strength f_c'. The response derivatives for each branch labeled in Fig. 8 are shown below.

Parabolic Ascending Branch ($\varepsilon < \varepsilon_{\min}$ and $\varepsilon > \varepsilon_c$)

Stress response

$$\sigma = f_c'(2\eta - \eta^2), \quad \eta = \frac{\varepsilon}{\epsilon_c} \tag{30}$$

Derivative of stress

$$\frac{\partial\sigma}{\partial f_c'} = (2\eta - \eta^2) + \frac{2f_c'}{\varepsilon_c}(1-\eta)\frac{\partial\varepsilon}{\partial f_c'} \tag{31}$$

Conditional derivative of stress

$$\left.\frac{\partial\sigma}{\partial f_c'}\right|_{\varepsilon} = (2\eta - \eta^2) \tag{32}$$

Linear Descending Branch ($\varepsilon < \varepsilon_{\min}$ and $\varepsilon < \varepsilon_c$)

Stress response

$$\sigma = f_c'\frac{\varepsilon_u - \varepsilon}{\epsilon_u - \varepsilon_c} \tag{33}$$

Derivative of stress

$$\frac{\partial\sigma}{\partial f_c'}=\frac{\varepsilon_u-\varepsilon}{\varepsilon_u-\varepsilon_c}-\frac{f_c'}{\varepsilon_u-\varepsilon_c}\frac{\partial\varepsilon}{\partial f_c'} \tag{34}$$

Conditional derivative of stress

$$\left.\frac{\partial\sigma}{\partial f_c'}\right|_{\varepsilon}=\frac{\varepsilon_u-\varepsilon}{\varepsilon_u-\varepsilon_c} \tag{35}$$

Unloading/Reloading Branch ($\varepsilon > \varepsilon_{\min}$)

Stress response

$$\sigma=\frac{\sigma_{\min}}{\varepsilon_{\min}}\varepsilon \tag{36}$$

Derivative of stress

$$\frac{\partial\sigma}{\partial f_c'}=\left(\frac{\varepsilon_{\min}\dfrac{\partial\sigma_{\min}}{\partial f_c'}-\dfrac{\partial\varepsilon_{\min}}{\partial f_c'}\sigma_{\min}}{\varepsilon_{\min}^2}\right)\varepsilon+\frac{\sigma_{\min}}{\varepsilon_{\min}}\frac{\partial\varepsilon}{\partial f_c'} \tag{37}$$

Conditional derivative of stress

$$\left.\frac{\partial\sigma}{\partial f_c'}\right|_{\varepsilon}=\left(\frac{\varepsilon_{\min}\dfrac{\partial\sigma_{\min}}{\partial f_c'}-\dfrac{\partial\varepsilon_{\min}}{\partial f_c'}\sigma_{\min}}{\varepsilon_{\min}^2}\right)\varepsilon \tag{38}$$

Given a strain ε and the history variables $\varepsilon_{\min}$ and $\sigma_{\min}$, the conditional derivative of the stress is computed from either Eq. (32), (35), or (38), depending on the active branch of the stress–strain relationship. This conditional derivative of stress contributes to the conditional derivative of the section forces in Eq. (27) during phase one of sensitivity computations.

The derivatives $\partial\sigma_{\min}/\partial f_c'$ and $\partial\varepsilon_{\min}/\partial f_c'$ are required to compute the conditional derivative of stress on branch 3, as seen in Eq. (38). As a result, these derivatives must be stored as sensitivity history variables. When on branches 1 and 2, $\partial\varepsilon_{\min}/\partial f_c'$ is set to $\partial\varepsilon/\partial f_c'$, as given by Eq. (28) during phase two of sensitivity computations. The derivative $\partial\sigma_{\min}/\partial f_c'$ is then computed by either Eq. (31) or (34). When on branch 3, $\varepsilon_{\min}$ does not change, so the sensitivity history variables $\partial\varepsilon_{\min}/\partial f_c'$ and $\partial\sigma_{\min}/\partial f_c'$ are not updated during unloading and reloading.

Notation

The following symbols are used in this paper:

- $\mathbf{a}_e$ = section deformation interpolation matrix;
- $\mathbf{a}_s$ = section compatibility matrix;
- $\mathbf{b}$ = section force interpolation matrix;
- $\mathbf{e}$ = section deformation vector;
- $\mathbf{f}$ = element flexibility matrix;
- $\mathbf{f}_s$ = section flexibility matrix;
- $\mathbf{K}_T$ = global tangent stiffness matrix;
- $\mathbf{k}$ = element stiffness matrix;
- $\mathbf{k}_s$ = section stiffness matrix;
- $\mathbf{P}_f$ = global external force vector;
- $\mathbf{P}_r$ = global resisting force vector;
- $\mathbf{q}$ = element basic force vector;
- $\mathbf{s}$ = section force vector;
- $\mathbf{U}$ = global displacement vector;
- $\mathbf{v}$ = element deformation vector;
- ε = material strain; parameter perturbation;
- θ = parameter;
- $\boldsymbol{\Theta}$ = parameter vector; and
- σ = material stress.

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