Discussion of "Evaluation of Force-Based Frame Element Response Sensitivity Formulations" by Michael H. Scott

January 2012, Vol. 138, No. 1, pp. 72–80.
DOI: 10.1061/(ASCE)ST.1943-541X.0000447

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Introduction

The author presents an analysis of two algorithms based on the direct differentiation method (DDM) for finite-element (FE) response sensitivity analysis applied to FE models containing force-based frame elements. The discussers would like to clarify some issues addressed in the paper and correct some misleading statements made in the paper.

Issue 1

The use of the terms consistent and inconsistent to characterize the two formulations compared in the paper is incongruent with the existing literature on FE response (e.g., Simo and Taylor 1985) and response sensitivity analysis (e.g., Vidal et al. 1991; Conte et al. 2003), as well as in numerical analysis. In particular, a FE response sensitivity algorithm is defined as consistent when it uses consistent differentiation of the underlying FE response algorithm; i.e., the use of algorithmic instead of continuum tangent operators and exact analytical differentiation of the discrete equilibrium, compatibility, and constitutive equations. In numerical analysis, consistency is a property of a numerical method with respect to a continuous problem, and it indicates that the discretized problem is a proper discretization of the continuous problem. For example, a consistent numerical method for partial differential equations ensures that the discretization becomes exact as the mesh size tends to zero (Δx_i, $\Delta t \rightarrow 0$) (Mattheij and Molenaar 2002). However, the attributes of consistent and inconsistent used in this paper relate to the propagation of the round-off error in finite precision arithmetic; i.e., they relate to an issue of numerical precision. Deviating from the terminology established in the literature is misleading and should be avoided.

Issue 2

As observed by the author, the two formulations compared in the paper are equivalent in exact arithmetic. Indeed, these two formulations are based on the exact analytical differentiation of the same discrete equations of equilibrium, compatibility, and constitutive models belonging to the force-based frame element. The mathematical equivalence of the two formulations can be shown analytically. Thus, both formulations are consistent based on the established terminology and the statement that only one of the formulations is consistent with the force-based response equations is incorrect.

Issue 3

A significant reduction of the condition number of matrix $\mathbf{S}$ in Eq. (23) of the paper can be obtained simply by premultiplying the trailing $N_{BF}(=3)$ rows of Eq. (23) by the current element stiffness matrix $\mathbf{k}_{(3\times3)}$; i.e., by rewriting Eq. (23) as

$$\begin{bmatrix} \mathbf{k}_{s1} & \mathbf{0} & \cdots & \mathbf{0} & -\mathbf{b}_1 \\ \mathbf{0} & \mathbf{k}_{s2} & \cdots & \mathbf{0} & -\mathbf{b}_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{k}_{sN} & -\mathbf{b}_N \\ w_1\mathbf{k}\mathbf{b}_1^T & w_2\mathbf{k}\mathbf{b}_2^T & \cdots & w_N\mathbf{k}\mathbf{b}_N^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \partial\mathbf{e}_1/\partial\theta \\ \partial\mathbf{e}_2/\partial\theta \\ \vdots \\ \partial\mathbf{e}_N/\partial\theta \\ \partial\mathbf{q}/\partial\theta \end{bmatrix} = \begin{bmatrix} -\partial\mathbf{s}_1/\partial\theta|_{\mathbf{e}_1} + (\partial\mathbf{b}_1/\partial\theta)\mathbf{q} + \partial\mathbf{s}_{p1}/\partial\theta \\ -\partial\mathbf{s}_2/\partial\theta|_{\mathbf{e}_2} + (\partial\mathbf{b}_2/\partial\theta)\mathbf{q} + \partial\mathbf{s}_{p2}/\partial\theta \\ \vdots \\ -\partial\mathbf{s}_N/\partial\theta|_{\mathbf{e}_N} + (\partial\mathbf{b}_N/\partial\theta)\mathbf{q} + \partial\mathbf{s}_{pN}/\partial\theta \\ \mathbf{k}\left\{\dfrac{\partial\mathbf{v}}{\partial\theta} - \sum_{i=1}^{N}\left[\left(\dfrac{\partial\mathbf{b}_i}{\partial\theta}\right)^T\mathbf{e}_i w_i + \mathbf{b}_i^T\mathbf{e}_i\dfrac{\partial w_i}{\partial\theta}\right]\right\} \end{bmatrix} \quad (1)$$

In this discussion, the formulation presented in Scott et al. (2004) is referred to as Method I, whereas the formulation developed by Conte et al. (2004) is referred to as Method II-Unscaled and as Method II-Scaled when Eq. (23) of the paper is replaced by Eq. (1). Through a simple algebraic manipulation, Eq. (1) has a condition number similar to that obtained from the equality constrained least-square approach presented in Appendix I of the paper.

Issue 4

Most of the conclusions drawn in the paper are based only on the condition numbers of the linear system of equations arising in the FE response sensitivity computation using the two algorithms considered. Although the condition number provides an upper bound of the potential error amplification in solving the linear system of equations $\mathbf{Ax} = \mathbf{b}$ (Cook et al. 2002), this upper bound usually drastically overestimates the actual error, unless vector $\mathbf{b}$ is proportional to the highest eigenvector (i.e., corresponding to the largest eigenvalue) of matrix $\mathbf{A}$ and its perturbation (or error) $\Delta\mathbf{b}$ is proportional to the lowest eigenvector (i.e., corresponding to the smallest eigenvalue) of matrix $\mathbf{A}$ (Haftka 1990). The joint occurrence of these two conditions is very unlikely for the linear systems of equations to be solved in Method II; i.e., Eq. (23) of the paper or Eq. (1). The perturbation analysis presented in the paper for a linear elastic simply supported prismatic beam provides a better estimate of the possible loss of accuracy. However, a perturbation analysis was not performed in the paper for the nonlinear inelastic example considered. Finally, a comparison of the FE response sensitivity results obtained using the two algorithms of interest would arguably provide a more direct and more precise approach to determine conclusively whether Method II suffers a significant relative loss of accuracy over Method I. However, no such comparison was provided in the paper to demonstrate decisively the claimed loss of accuracy for Method II.

Issues 3 and 4 are illustrated using the same two examples presented in the paper (i.e., linear elastic simply supported prismatic

J. Struct. Eng. 2013.139:1095-1096.

beam and nonlinear inelastic RC column) analyzed using *FedeasLab* (Filippou and Constantinides 2004) in which Method I, Method II-Unscaled, and Method II-Scaled were implemented by the discussers. Fig. 1 shows the condition numbers corresponding to Method I, Method II-Unscaled, and Method II-Scaled for a linear-elastic prismatic frame element with varying radii of gyration r, and $E = 2.0 \times 10^8$ kPa, $A = 0.01$ m^2, and $L = 8.0$ m. Fig. 2 plots the results of a perturbation analysis performed for Method II (both unscaled and scaled) according to Eq. (31) of the paper. It is observed that, for this example, the condition number corresponding to Method II-Scaled decreases from $[10^{11}-10^{13}]$ for Method II-Unscaled to $[10^6-10^7]$, and the loss of accuracy for Method II-Scaled is negligible (i.e., zero digit lost for each of the 13 equations considered, where $NN_{SR} + N_{BF} = 5 \times 2 + 3 = 13$, in which $N = 5$ is the number of Gauss-Lobatto points, $N_{SR} = 2$ is the number of section forces, and $N_{BF} = 3$ is the number of element basic forces). Fig. 3 shows the condition numbers for Method I, Method II-Unscaled, and Method II-Scaled as a function of the horizontal tip displacement, Δ, of the nonlinear RC cantilever column. Fig. 4 shows the maximum number of digits lost over all 11 equations ($NN_{SR} + N_{BF} = 4 \times 2 + 3 = 11$) as a function of Δ for both Method II-Unscaled and Method II-Scaled. These results were obtained using perturbation analysis. Also in this case, Method II-Scaled results in significantly smaller condition numbers than Method II-Unscaled and in a negligible loss of accuracy. The inset in Fig. 4 provides the pushover curve relating the horizontal force, P, to Δ for the same RC cantilever column. Both the FE response (see the inset in Fig. 4) and condition numbers (see Fig. 3) are very similar to those presented in the paper. The small differences observed are most likely a result of the use of material constitutive models and fiber section discretization that are different from those adopted by the author to model the same RC column. Finally, Fig. 5 plots the normalized sensitivity of Δ to the yield strength, f_y, of the reinforcement steel (i.e., $\partial\Delta/\partial f_y \times f_y/\Delta$) as a function of Δ for the three methods considered in this discussion. It was verified that the three FE response sensitivity curves coincide in double precision. A zoom view of these sensitivity results is also provided in the inset of Fig. 5.

It is worth mentioning that the discussers have utilized Method II-Unscaled in numerous studies for various application examples

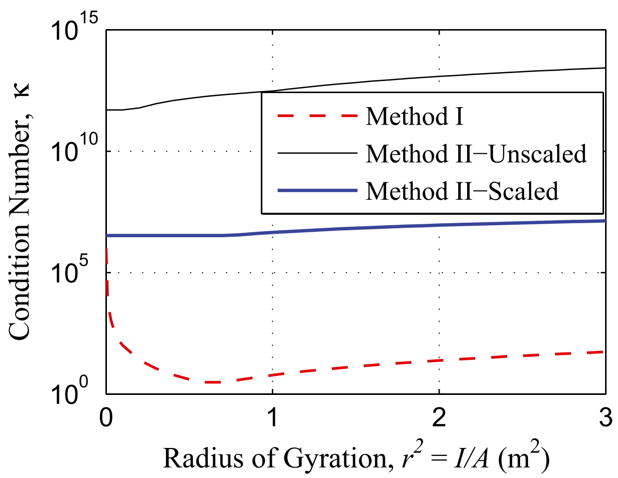

Fig. 1. Condition numbers of various force-based response sensitivity formulations for a linear elastic prismatic frame element with varying radii of gyration and $E = 2.0 \times 10^8$ kPa, $A = 0.01$ m^2, and $L = 8.0$ m

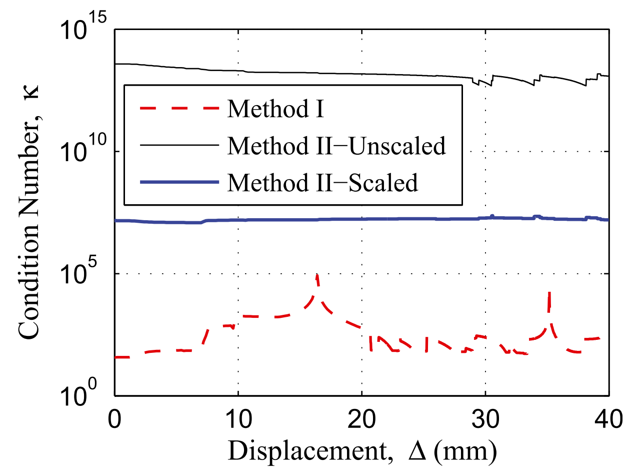

Fig. 3. RC column example: condition numbers of various force-based response sensitivity formulations

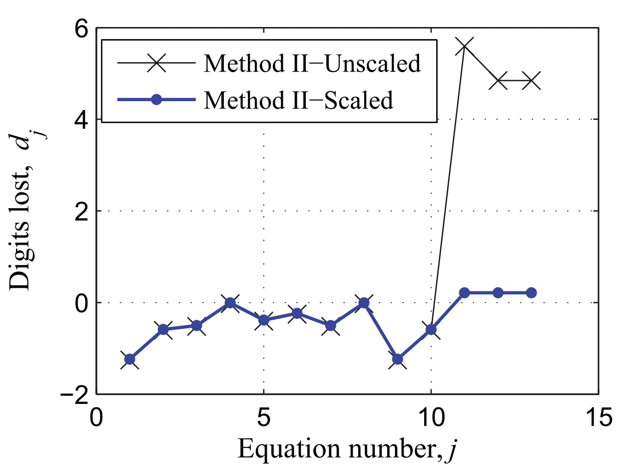

Fig. 2. Digits lost in Method II estimated using perturbation analysis: linear elastic prismatic frame element with $E = 2.0 \times 10^8$ kPa; $A = 0.01$ m^2; $L = 8.0$ m; $r^2 = 0.04$ m^2; number of integration points $N = 5$; sensitivity parameter L = element length

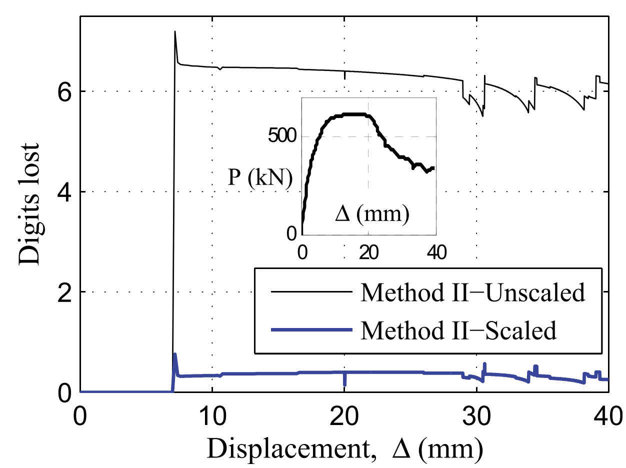

Fig. 4. RC column example: maximum number of digits lost over all equations using perturbation analysis (inset: load-displacement response)

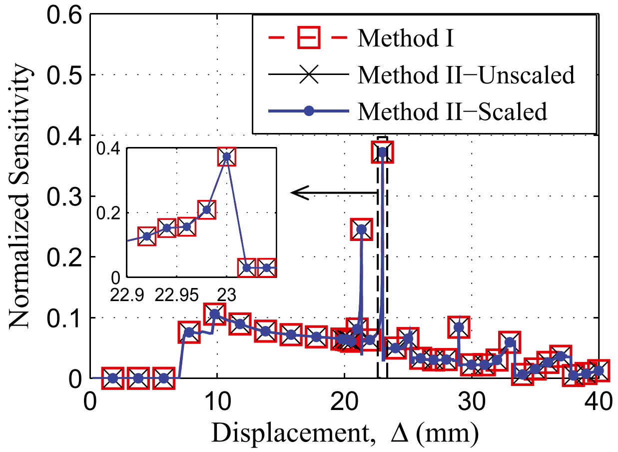

Fig. 5. RC column example: normalized sensitivity of Δ with respect to f_y computed using various FE response sensitivity formulations for the force-based element (inset: zoom view)

and have not encountered any case in which the potential loss of accuracy predicted by this paper has actually taken place.

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Closure to "Evaluation of Force-Based Frame Element Response Sensitivity Formulations" by Michael H. Scott

January 2012, Vol. 138, No. 1, pp. 72–80.
DOI: 10.1061/(ASCE)ST.1943-541X.0000447

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The author would like to thank the discussers for their interest in the original paper and for addressing numerical issues related to response sensitivity analysis using force-based frame finite elements. The discussers introduced terminology for Method I, Method II-unscaled, and Method II-scaled that will be used herein. Method I refers to the response sensitivity formulation presented by Scott et al. (2004), and Method II-unscaled refers to the formulation of Conte et al. (2004). The numerical algorithms for these two formulations were compared in the original paper, and Method II-scaled was presented in the discussion as a modified version of the Conte et al. (2004) formulation. Responses to the four issues raised by the discussers follow.

Issue 1

The discussers noted that the consistent/inconsistent terminology used with respect to the two formulations compared in the original paper is incongruent with existing finite-element literature. In the paper, Method I was termed consistent, because the element response sensitivity is computed using only objects that are formed in computing the ordinary response. This is in contrast to Method II-unscaled, which used an aggregation of these objects [Eq. (23)] not seen in the ordinary response, resulting in its label as inconsistent.

Issue 2

As noted by the discussers, the mathematical equivalence of the two formulations can be shown analytically. This was shown in the original paper; however, the discussers once again bring up the semantics of the consistent terminology with this issue. As pointed out in the response to Issue 1, consistency in the original paper referred to the similarity of the response sensitivity algorithm with the algorithm required to compute the ordinary response.

Issue 3

The discussers multiply the trailing N_{BF} rows of Eq. (23) in the original paper by the basic stiffness matrix, **k**, leading to Eq. (1) in the discussion, or Method II-scaled. This reduces the condition number of the matrix originally proposed in Method II-unscaled to the order of the equality-constrained least-squares approach presented in Appendix I of the original paper. This is a welcome modification to Method II-unscaled; however, the condition number of Method II-scaled remains much higher than that associated with Method I.

Issue 4

The discussers state that most of the conclusions drawn in the original paper are based only on the condition number of the matrix in Eq. (23), which can severely overestimate the actual error. To support this statement, the discussers cite an error analysis by Haftka (1990) based on the proportionality of the right-hand side vector and its perturbation to the eigenvectors that correspond to the highest and lowest eigenvalues of the left-hand side matrix.

The analysis by Haftka (1990) was based on linear systems of equations with a symmetric positive definite (SPD) left-hand side matrix. An SPD matrix has eigenvalues that are all real and positive; however, neither of the matrices in question for Method II-unscaled [Eq. (23) of the original paper] and Method II-scaled [Eq. (1) of the discussion] is SPD. Furthermore, both of these matrices can have complex eigenvalues under realistic scenarios where the structural stiffness matrix is SPD.

Consider a force-based frame element of length $L = 8$ m with five Gauss-Lobatto points, where the axial and flexural responses are uncoupled at the section level with $E = 2.0e5$ MPa, $A = 0.01$ m^2, and $I = 4e\text{-}4$ m^4. It is straightforward to show that when flexural yielding occurs at any one of the five sections and the flexural stiffness of that section is reduced to $0.001EI$ in an attempt to simulate perfectly plastic behavior, the basic stiffness of the element remains SPD. However, the resulting nonsymmetric matrix of Method II-scaled computed at this element state possesses one pair of complex eigenvalues ($0.04199 \pm 0.2797i$ when yielding occurs at either end of the element). When the same analysis is repeated with E defined equivalently as 200 GPa instead of 2.0e5 MPa, three pairs of complex eigenvalues appear for the Method II-scaled matrix ($1 \pm i$, $0.04 \pm 0.28i$, and $0.002112 \pm 0.008939i$). The associated eigenvectors are also complex.

The discussers show by perturbation analysis that there is not a significant loss of accuracy for Method II-scaled compared with Method I. However, it is unclear how the results presented by Haftka (1990) relate to Method II-scaled when the matrix possesses complex eigenvectors, because proportionality of the right-hand side vector in Eq. (1) of the discussion to a complex vector is not possible. The generalization of algorithmic analyses based on SPD matrices to problems with nonsymmetric matrices is misleading and should be avoided.

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