Solving a Nonlinear Equation: Newton's Method

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Newton's Method for Solving a Nonlinear Equation

Scenario: You have an equation \( y(x) = 0 \), that is, there is a certain \( x \) value that makes the function equal to zero. But, the equation is too messy to solve directly. You do have an initial guess of the correct value of \( x \). It is close, but it is wrong.

For example, solve this equation for \( x \):

\[
y(x) = \cos^3 x + \log_{10} x = 0
\]

Starting with an initial guess of \( x = 6 \)

You can take the \( x \) you have, \( x_{\text{have}} \), and plug it into the equation to produce \( y_{\text{have}} \) and thus see how close you are to \( y = 0 \). But now what?

From calculus, we know that:

\[
\frac{dy}{dx} \approx \Delta y \quad \text{or} \quad \frac{dy}{dx} \Delta x = \Delta y
\]

So that:

\[
\frac{dy}{dx} \Delta x = \Delta y = y_{\text{new}} - y_{\text{have}} = 0 - y_{\text{have}}
\]

which gives us:

\[
\Delta x = \frac{-y_{\text{have}}}{\frac{dy}{dx}}
\]

We will use that to find the next value of \( x \) to try, and then repeat the process:

\[
x_{\text{have}} = x_{\text{have}} + \Delta x = x_{\text{have}} + \frac{-y_{\text{have}}}{\frac{dy}{dx}}
\]

\[
y_{\text{have}} = y(x_{\text{have}})
\]
Here's what is really going on

\[ y = \cos^2 x + \log_{10} x = 0 \]
\[ \frac{dy}{dx} = -3\sin x \cos^2 x + \frac{1}{x \ln(10)} \]

<table>
<thead>
<tr>
<th>( x_{\text{next}} )</th>
<th>( y_{\text{next}} )</th>
<th>( \Delta y )</th>
<th>( \Delta x )</th>
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</tbody>
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What would have happened if we had started with \( x = 2.75 \)?

\[ x = 2.75000 \]
\[ y = -0.35033 \]
\[ \Delta y = -0.82027 \]
\[ \Delta x = 2.32291 \]

<table>
<thead>
<tr>
<th>( x_{\text{next}} )</th>
<th>( y_{\text{next}} )</th>
<th>( \Delta y )</th>
<th>( \Delta x )</th>
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What would have happened if we had started with $x=0.55$?

A Collision Detection Example

Let’s say we have a nonlinear surface. How close is the point $(3,1)$ to that surface?

Using our friend, the dot product:

$$(P_x - Q_x, P_y - Q_y) \cdot \text{slope} = 0$$

where the vector slope is:

$$\text{slope} = (dx, dy) = (1, \frac{dy}{dx}) = (1, \frac{d\sin x}{dx}) = (1, \cos x)$$

substituting for $Q_x$, $Q_y$, and the slope:

$$f(x) = (P_x - x, P_y - \sin x) \cdot (1, \cos x) = 0$$

and expanding:

$$f(x) = (P_x - x) + \cos x \cdot (P_y - \sin x) = 0$$

Note that in this case, we are solving $f(x) = 0$, not $y(x) = 0$!
A Collision Detection Problem Example

\[ f(x) = (P_x - x) + \cos x \ast (P_x - \sin x) = 0 \]

<table>
<thead>
<tr>
<th>xhave</th>
<th>yhave</th>
<th>fhave</th>
<th>dfdx</th>
<th>xnext</th>
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\[ \text{dist} = \sqrt{(3 - 2.59234)^2 + (1 - 0.52205)^2} \]

\[ \text{dist} = 0.62819 \]