Periodic Steady-State Analysis of Oscillators with a Specified Oscillation Frequency

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***Abstract*— In this paper a modification of the time-domain periodic steady-state analysis for oscillators is presented. The proposed analysis finds the value of a circuit parameter that results in the circuit oscillating at a desired frequency. This analysis is based on the steady-state analysis for voltage and current controlled oscillators, i.e., replacing the oscillation period by a circuit parameter in the list of unknowns. A generalized formulation that can handle a control voltage or current, as well as any frequency-tuning circuit parameter, such as a tank capacitor or device geometry is developed in this paper.**

I. Introduction

Oscillators are commonly used in digital systems for clock generation, and in radio-frequency communication front-ends for up- and down-conversion. System specifications include clock frequency for digital systems, and channel frequency bands for radio-frequency systems. In these applications, the oscillator fundamental frequency f_0 is specified during the design phase. However, the conventional periodic steady-state (PSS) analysis [2], [3] treats the oscillation period $T = 1/f_0$ as an unknown.

Consider the problem of finding the control voltage for a voltage-controlled oscillator (VCO) such that the circuit oscillates at the specified frequency. Most existing analysis tools can offer only a brute-force search-based approach, which requires running conventional PSS analyses at a number of control voltage values. A more efficient and elegant solution to this problem is the PSS analysis for voltage and current controlled oscillators [1]. This method treats the control voltage or current as an unknown, while the oscillation frequency is specified as a parameter.

However, the control voltage is not an appropriate frequency-tuning circuit parameter for use in the design phase. In the operation of a phase-locked loop (PLL), the VCO frequency is adjusted by the control voltage in closed-loop, and the range of the control voltage is determined by the charge pump design [7]. Therefore, in designing an open-loop VCO, the control voltage should not be used as the frequency-tuning parameter, as this affects the tuning range. Instead, the tank capacitor or the delay stage device size can be used as a frequency-tuning parameter in LC or ring oscillators, respectively. In crystal oscillators, the tuning capacitor should be changed to adjust the frequency.

The formulation described in [1] works specifically with a control voltage or current as the frequency tuning parameter. In this paper, a generalized formulation of the PSS analysis for oscillators is presented. We call it PSS analysis with a specified oscillation frequency. Our formulation is capable of working with a control voltage or current, as well as any circuit parameter that affects the oscillation frequency.

In Section II we review the concept of the steady-state analysis with a specified frequency. A general mathematical description for oscillators is provided. Based on this description the difference between the conventional and proposed PSS analyses in terms of the problem formulation and performance is explained. In Section III a discrete-time oscillator representation suitable for computer simulation [4] is presented. Based on this model a finite difference method for PSS analysis with a specified frequency is presented. In Section IV, simulation results for LC and ring oscillator circuits are given. It is shown that the results obtained by the proposed PSS analysis are consistent with the results of a conventional PSS analysis. Finally, the paper is concluded in Section V.

II. Theoretical Formulation

In this section the concept of oscillator PSS analysis with a specified oscillation frequency is reviewed. The proposed PSS formulation is compared to the conventional one, based on a general continuous-time mathematical representation for oscillators.

A. Continuous-Time Oscillator Equations

Any nonlinear oscillator circuit can be modeled as a set of m differential-algebraic equations (DAEs) given by

$$\dot{q}\big(x(t), \gamma_{f_0}\big) + f\big(x(t), \gamma_{f_0}\big) + b(\gamma_{f_0}) = 0 \tag{1}$$

where

$$\begin{aligned} t \in \mathbb{R} &: \text{time, independent variable,} \\ \gamma_{f_0} \in \mathbb{R} &: \text{oscillator circuit parameter,} \\ x : \mathbb{R} \to \mathbb{R}^m &: \text{oscillator state variables,} \\ q : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^m &: \text{contribution of reactive components,} \\ f : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^m &: \text{contribution of resistive components,} \\ b : \mathbb{R} \to \mathbb{R}^m &: \text{independent sources.} \end{aligned}$$

The T-periodic solution $x(t)$ of DAEs in (1) is called the PSS solution and satisfies $x(t) = x(t+T)$. This periodicity constraint can be expressed as

$$x(0) = x(T) \tag{2}$$

Note that if $x(t)$ is a PSS solution, then $x(t+\Delta t)$, $\forall \Delta t$ is also a valid PSS solution. A unique isolated solution can be selected by imposing a phase condition

$$\varphi\big(x(0)\big) = 0, \quad \varphi : \mathbb{R}^m \rightarrow \mathbb{R} \tag{3}$$

One possible phase condition is to let a component of $x(0)$ be a fixed value.

The oscillator PSS is uniquely defined by the system of (1), (2), and (3), resulting in the continuous-time equations for the oscillator in the steady-state

$$\begin{cases} \dot{q}\big(x(t), \gamma_{f_0}\big) + f\big(x(t), \gamma_{f_0}\big) + b(\gamma_{f_0}) = 0 \\ x(0) = x(T) \\ \varphi\big(x(0)\big) = 0 \end{cases} \tag{4}$$

This is a periodic boundary value problem (BVP), a special case of a two-point BVP [5].

B. PSS Analysis: Conventional vs Specified Frequency

A conventional PSS analysis computes the periodic waveform $x(t)$ and the oscillation period T, for a given parameter γ_{f_0}

$$\gamma_{f_0} \rightarrow \boxed{\text{Eq. (4)}} \rightarrow \{x(t), T\} \tag{5}$$

Here, the period T is one of the unknowns, and the circuit parameter γ_{f_0} is a parameter of the oscillator equations.

The idea behind the proposed PSS analysis is to swap the role of T for the role of γ_{f_0}, i.e., to introduce γ_{f_0} as an unknown, and treat the period T as a known parameter. The objective of the PSS analysis with a specified oscillation frequency is to find the value of the circuit parameter γ_{f_0}, and the periodic waveform $x(t)$ for a given oscillation period T

$$T \rightarrow \boxed{\text{Eq. (4)}} \rightarrow \{x(t), \gamma_{f_0}\} \tag{6}$$

The computational cost of the PSS analysis with a specified frequency is comparable to that of the conventional PSS analysis. However, solving the problem in (6) is computationally more efficient than using the conventional analysis methods. The brute force approach to the problem in (6) is to alter γ_{f_0} (manually or automatically), while monitoring T with a conventional PSS analysis

$$\begin{aligned} \gamma_{f_0}^{(0)} &\rightarrow \boxed{\text{Eq. (4)}} \rightarrow \{x(t)^{(0)}, T^{(0)}\} \\ \gamma_{f_0}^{(1)} &\rightarrow \boxed{\text{Eq. (4)}} \rightarrow \{x(t)^{(1)}, T^{(1)}\} \\ &\vdots \\ \gamma_{f_0}^{(N)} &\rightarrow \boxed{\text{Eq. (4)}} \rightarrow \{x(t)^{(N)}, T^{(N)} = T_{target}\} \end{aligned}$$

Therefore, the proposed method for solving (6) is about N times faster than the brute-force approach.

In contrast to [1], our formulation is general, as it works with a control voltage $b(\gamma_{f_0} \equiv V_{ctrl})$ or a control current $b(\gamma_{f_0} \equiv I_{ctrl})$ as well as any circuit parameter that affects the oscillation frequency, such as a tank capacitor $q\big(x(t), \gamma_{f_0} \equiv C_{tank}\big)$, MOSFET width $f\big(x(t), \gamma_{f_0} \equiv W\big)$, $q\big(x(t), \gamma_{f_0} \equiv W\big)$, etc.

III. Numerical Methods for Steady-State Analysis with a Specified Oscillation Frequency

In this section numerical methods for computing the oscillator steady-state with a specified oscillation frequency based on a discrete-time oscillator description are presented.

A. Discrete-Time Oscillator Equations

Analysis of nonlinear oscillators using the continuous-time representation (4) is impractical. For numerical time-domain PSS analysis, time is discretized, and the time-derivative operator is replaced by a finite-difference approximation. As an example, using uniformly spaced timepoints $t_i = ih$, $i \in \mathbb{N}$ and applying the backward Euler method, a simple discrete counterpart of (4) is

$$\begin{cases} \frac{1}{h}(q_i - q_{i-1}) + f_i + b = 0, \ i = 1, \ldots, n \\ x_0 = x_n \\ \varphi(x_0) = 0 \end{cases} \tag{7}$$

where

$$\begin{aligned} q_i &= q(x_i, \gamma_{f_0}), & x_i &\equiv x(t_i), \\ f_i &= f(x_i, \gamma_{f_0}), & t_i &= ih, \\ b &= b(\gamma_{f_0}), & h &= h(T) = T/n. \end{aligned}$$

The discrete-time description in (7) is a system of $nm+m+1$ nonlinear algebraic equations. The equations are written in terms of $(n+1)m$ PSS waveform samples x_i, $i = 0, \ldots, n$, the circuit parameter γ_{f_0}, and the oscillation period T. As proposed in Section II-B the circuit parameter γ_{f_0} and x_i are the unknowns, and the oscillation period T is a known parameter.

B. Finite Difference Method

The equations in (7) can be written in the following form

$$\begin{bmatrix} \frac{1}{h}(q_1 - q_n \quad) + f_1 + b \\ \frac{1}{h}(q_2 - q_1 \quad) + f_2 + b \\ \vdots \\ \frac{1}{h}(q_n - q_{n-1}) + f_n + b \\ \varphi(x_n) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \tag{8}$$

Note that the periodicity constraint $x_0 = x_n$ is not explicitly present in the above system. The periodicity constraint equations were used to eliminate x_0 from the list of unknowns. The remaining $nm+1$ equations represent a finite difference formulation of the proposed PSS analysis. Denoting the left hand side of (8) by $F_{fd}(x_1, \ldots, x_n, \gamma_{f_0})$, $F_{fd} : \underbrace{\mathbb{R}^m \times \ldots \times \mathbb{R}^m}_{n} \times \mathbb{R} \rightarrow \mathbb{R}^{nm+1}$, we rewrite the finite difference system as

$$F_{fd}(x_1, \ldots, x_n, \gamma_{f_0}) = 0 \tag{9}$$

This system of nonlinear equations can be solved using the Newton-Raphson iteration

$$J_{fd}\big(X^{(k)}\big)\left[X^{(k+1)} - X^{(k)}\right] = -F_{fd}\big(X^{(k)}\big) \tag{10}$$

$$J_{fd}(x_1,\ldots,x_n,\gamma_{f_0}) = \begin{bmatrix} \frac{1}{h}C_1+G_1 & & & -\frac{1}{h}C_n & \frac{1}{h}\left(\frac{\partial q_1}{\partial\gamma_{f_0}}-\frac{\partial q_n}{\partial\gamma_{f_0}}\right)+\frac{\partial f_1}{\partial\gamma_{f_0}}+\frac{db}{d\gamma_{f_0}} \\ -\frac{1}{h}C_1 & \frac{1}{h}C_2+G_2 & & & \frac{1}{h}\left(\frac{\partial q_2}{\partial\gamma_{f_0}}-\frac{\partial q_1}{\partial\gamma_{f_0}}\right)+\frac{\partial f_2}{\partial\gamma_{f_0}}+\frac{db}{d\gamma_{f_0}} \\ & \ddots & \ddots & & \vdots \\ & & -\frac{1}{h}C_{n-1} & \frac{1}{h}C_n+G_n & \frac{1}{h}\left(\frac{\partial q_n}{\partial\gamma_{f_0}}-\frac{\partial q_{n-1}}{\partial\gamma_{f_0}}\right)+\frac{\partial f_n}{\partial\gamma_{f_0}}+\frac{db}{d\gamma_{f_0}} \\ 0 & \cdots & 0 & \left.\frac{\partial\varphi(x)}{\partial x}\right|_{x_n} & 0 \end{bmatrix} \quad (11)$$

where k is the iteration index, $X = [x_1,\ldots,x_n,\gamma_{f_0}]^T$ is the vector of the finite difference unknowns, $J_{fd}(x_1,\ldots,x_n,\gamma_{f_0}) = \partial F_{fd}/\partial X$ is the Jacobian matrix, $J_{fd} : \underbrace{\mathbb{R}^m\times\ldots\times\mathbb{R}^m}_{n}\times\mathbb{R} \rightarrow \mathbb{R}^{(nm+1)\times(nm+1)}$, given by (11). The Jacobian matrix is defined in terms of C_i and G_i, the capacitance and conductance matrices

$$C_i = \frac{\partial q_i}{\partial x_i} = \left.\frac{\partial q(x,\gamma_{f_0})}{\partial x}\right|_{x_i},\ C_i : \mathbb{R}^m\times\mathbb{R}\rightarrow\mathbb{R}^{m\times m} \quad (12)$$

$$G_i = \frac{\partial f_i}{\partial x_i} = \left.\frac{\partial f(x,\gamma_{f_0})}{\partial x}\right|_{x_i},\ G_i : \mathbb{R}^m\times\mathbb{R}\rightarrow\mathbb{R}^{m\times m} \quad (13)$$

For large problems, fast preconditioned iterative methods [1], [6] are employed to solve the linear system in (10).

The last column of the Jacobian matrix in (11) requires $\partial q/\gamma_{f_0}$, $\partial f/\gamma_{f_0}$, and $db/d\gamma_{f_0}$. These derivatives with respect to the frequency-tuning parameter can be obtained analytically or numerically from device models.

The Newton-Raphson method is a method with local convergence, and therefore the initial guess must be close enough to the solution. Particularly, the initial guess for the circuit parameter γ_{f_0} must be such that the circuit oscillates. Even with a good initial guess and existence of a solution, it is possible that the circuit stops being an oscillator in the middle of the Newton-Raphson iterative loop. Such a situation is untypical for the conventional PSS analysis, and it requires special treatment to recover, such as roll-back and damping. There may be no solution to the problem in (6), which means that the circuit can not oscillate at the desired frequency, independent of the parameter value.

IV. Examples and Results

We have implemented the PSS analysis with a specified oscillation frequency in our Matlab-based circuit simulator, and Berkeley Design Automation's RF FastSPICE simulator. In this section, simulation results for a cross-coupled LC-tank oscillator and a three-stage ring oscillator are presented. The simulation is done in two steps. First, a circuit is simulated using the PSS analysis with a specified frequency

$$T_{\text{SF}} \rightarrow \boxed{\text{PSS specified frequency}} \rightarrow \{x_{i_{\text{SF}}}, \gamma^*_{f_0}\}$$

Second, the conventional PSS analysis is used with the circuit parameter value $\gamma^*_{f_0}$ found in the first step

$$\gamma^*_{f_0} \rightarrow \boxed{\text{conventional PSS}} \rightarrow \{x_{i_{\text{C}}}, T_{\text{C}}\}$$

The two sets of results, as will be shown, are consistent with each other, i.e., $T_{\text{SF}} \approx T_{\text{C}}$, and $x_{i_{\text{SF}}} \approx x_{i_{\text{C}}}$. The mismatch between T_{SF} and T_{C} will be shown to be within the relative simulation tolerance of $\epsilon_{rel} = 10^{-3}$.

A. Cross-coupled LC-tank Oscillator

Consider an NMOS cross-coupled LC-tank oscillator shown in Figure 1. The tank parameters $L_1 = L_2 =$ 1nH, and

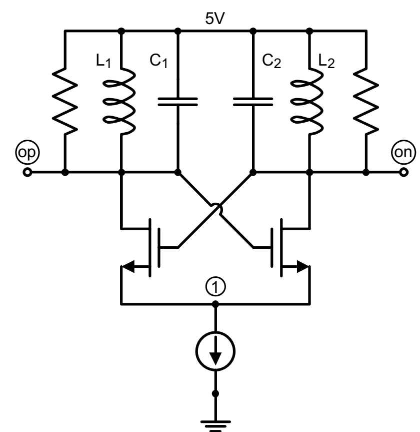

Fig. 1. Schematic of an NMOS cross-coupled LC-tank oscillator.

$C_1 = C_2 =$ 1pF define the oscillation period $T =$ 200.4ps. Now we find the value of the tank capacitors $\gamma_{f_0} \equiv C_1 = C_2$ such that the oscillation period T_{SF} is 150ps. The circuit was analyzed by a PSS analysis with a specified frequency

$$T_{\text{SF}} = \underset{\sim\sim\sim}{150\text{ps}} \rightarrow \boxed{\text{PSS specified frequency}} \rightarrow \{x_{i_{\text{SF}}}, \gamma^*_{f_0}\}$$

The solution $\gamma^*_{f_0} =$ 560.32fF was verified by the conventional PSS analysis

$$\gamma^*_{f_0} \rightarrow \boxed{\text{conventional PSS}} \rightarrow \{x_{i_{\text{C}}}, T_{\text{C}} = \underset{\sim\sim\sim\sim\sim\sim}{149.99999993\text{ps}}\}$$

As can be seen, T_{SF}, and T_{C} match within the simulation error tolerance of $\epsilon_{rel} = 10^{-3}$. The two families of PSS waveforms $x_{i_{\text{SF}}}$, and $x_{i_{\text{C}}}$ in Figure 2 are also in good agreement.

B. Three-Stage Ring Oscillator

The three-stage ring oscillator in Figure 3 is a chain of three identical inverters connected in a loop. The oscillation frequency is set by the gain and delay of the inverter stage. One way to change the oscillation frequency is to alter the MOSFET sizes. Let the width of the p-channel devices be

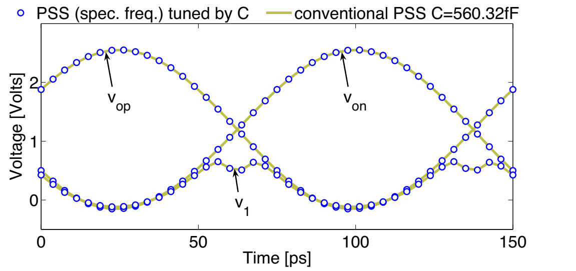

Fig. 2. PSS waveforms of the NMOS cross-coupled LC-tank oscillator.

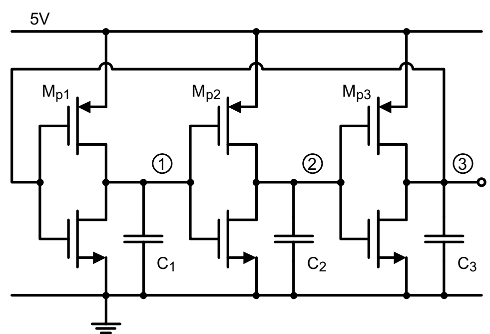

Fig. 3. Schematic of a three-stage ring oscillator.

$\gamma_{f_0} \equiv W_{M_{p1}} = W_{M_{p2}} = W_{M_{p3}}$ such that the oscillation period T_{SF} is 2ns. The circuit was analyzed by the PSS analysis with a specified oscillation frequency starting from an initial state of 19.463μm, corresponding to $T = 2.5$ns

$$T_{\text{SF}} = 2.000\text{ns} \rightarrow \boxed{\text{PSS specified frequency}} \rightarrow \{x_{i_{\text{SF}}}, \gamma^*_{f_0}\}$$

The solution $\gamma^*_{f_0} = 30.477\mu$m was verified by the conventional PSS analysis

$$\gamma^*_{f_0} \rightarrow \boxed{\text{conventional PSS}} \rightarrow \{x_{i_{\text{C}}}, T_{\text{C}} = 1.9999998\text{ns}\}$$

Oscillation periods T_{SF} and T_{C}, as well as the two families of PSS waveforms $x_{i_{\text{SF}}}$ and $x_{i_{\text{C}}}$ in Figure 4 are in good agreement.

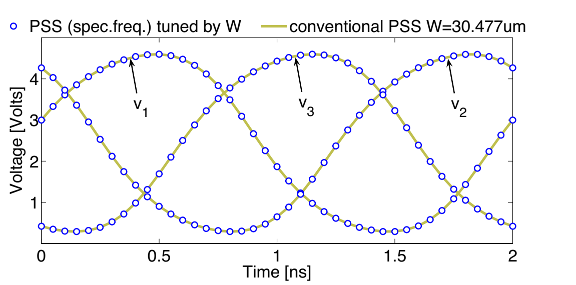

Fig. 4. PSS waveforms of the three-stage ring oscillator.

Another way to change the oscillation frequency is to alter the size of capacitors. Let us find the capacitance value $\gamma_{f_0} \equiv C_1 = C_2 = C_3$ such that the oscillation period T_{SF} is 2ns. The circuit was analyzed by the PSS analysis with a specified oscillation frequency starting from an initial state of 10fF, corresponding to $T = 2.5$ns

$$T_{\text{SF}} = 2\text{ns} \rightarrow \boxed{\text{PSS specified frequency}} \rightarrow \{x_{i_{\text{SF}}}, \gamma^*_{f_0}\}$$

The solution $\gamma^*_{f_0} = 7.9559$fF was verified by the conventional PSS analysis

$$\gamma^*_{f_0} \rightarrow \boxed{\text{conventional PSS}} \rightarrow \{x_{i_{\text{C}}}, T_{\text{C}} = 1.99999999993\text{ns}\}$$

Oscillation periods T_{SF} and T_{C}, as well as the two families of PSS waveforms $x_{i_{\text{SF}}}$ and $x_{i_{\text{C}}}$ in Figure 5 are in good agreement.

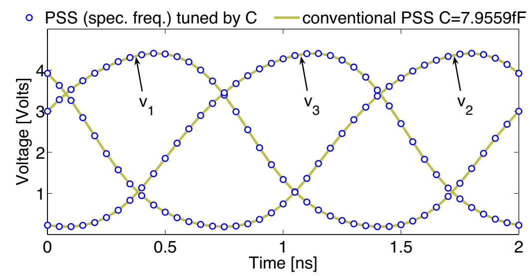

Fig. 5. PSS waveforms of the three-stage ring oscillator.

V. Conclusion

In many applications, the oscillator fundamental frequency is a given design specification. We have presented a general formulation and a numerical method for oscillator PSS analysis with a specified oscillation frequency. This analysis finds the value of a circuit parameter that results in the circuit oscillating at the desired frequency. The proposed formulation is general and handles any frequency-tuning circuit parameter. Our method is considerably faster than brute-force search-based approaches that employ the conventional PSS analysis. Simulation results show that the proposed PSS analysis is in good agreement with the conventional PSS analysis.

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