CS515: Analysis of Algorithms, Fall 2017

Final Exam

- Solve **at most 4** problems of the following 5 problems.
- *I don't know policy:* you may write "I don't know" *and nothing else* to answer a question and receive 25 percent of the total points for that problem whereas a completely wrong answer will receive zero.
- You can use algorithms in class or in the assignments as black boxes.

Problem 1. [**25 pts.**] Let (S, T) and (S', T') be two s, t-minimum cuts of a graph G. Show that $(S \cap S', T \cup T')$ and $(S \cup S', T \cap T')$ are also minimum cuts.

Problem 2. [**25 pts.**] Consider the following sorting algorithm:

```
RandomSort A[1..n]
  for i ← 2 to n in random order do
    k ← i
    while (k > 1) and (A[k] < A[k − 1]) do
      swap(A[k], A[k − 1])          (∗)
      k ← k − 1
    end while
  end for
```

(a) In the worst case, exactly how many times does RandomSort execute line $(*)$?

(b) What is the exact expected number of executions of line $(*)$ during the nth iteration of the for loop?

(c) What is the *exact* expected number of executions of line $(*)$ during the entire algorithm?

Problem 3. [**25 pts.**]

(a) A *doubtful* walk on integers starts at zero. At each step it flips a coin: upon a head it moves $+1$, and upon a tail it does not move.

  (i) What is the expected number of coin flips for a doubtful walk to take one step?

  (ii) What is the expected number of coin flips to reach n?

(b) A *very doubtful* walk on integers starts at zero. At each step it flips a coin: upon a head it moves $+1$, and upon a tail it moves -1. What is the expected number of coin flips for a very doubtful walk to reach 1?

Problem 4. A square matrix is called *interesting* if it is composed of non-negative real numbers such that the sum of the elements of every row is one, and the sum of elements of every columns is one. Given a subset of the entries of a metrix, we want to decide if it is possible to complete the matrix into an interesting matrix. For example, the partial metrix

$$\begin{bmatrix} 0.25 & 0.25 & - \\ - & - & - \\ - & - & 0.3 \end{bmatrix},$$

can be completed to the interesting matrix

$$\begin{bmatrix} 0.25 & 0.25 & 0.5 \\ 0.75 & 0.05 & 0.2 \\ 0 & 0.7 & 0.3 \end{bmatrix}.$$

Describe an algorithm that decides if a partial matrix can be completed to a an interesting matrix (Hint: find a reduction to the maximum flow problem.)

Problem 5. [**25 pts.**] Consider the following linear programming.

$$\begin{aligned} \text{maximize} \quad & 2x_1 + 3x_2 \\ \text{subject to} \quad & x_1 + 2x_2 \leq 4 \\ & x_1 + x_2 \leq 3 \\ & x_1, x_2 \geq 0 \end{aligned}$$

(a) Show all constraints and the feasible region by drawing a figure.

(b) Show the direction of the objective function.

(c) List all feasible vertices, and locally optimal vertices.

(d) Find the dual of the LP described above.

(e) Show the constraints of the dual LP and its feasible region by drawing a figure.

(b) Show the direction of the objective function in the dual.

(c) List all feasible vertices, and locally optimal vertices in the dual.