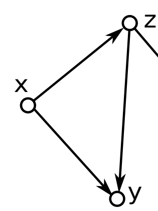
CS420/520: Graph Theory with Applications to CS, Winter 2016

Midterm

Tue, Feb/16/16

(1) This exam has 11 problems, each worth 10 points. (2) This is a closed book exam, but ONE sheet of notes is allowed. (3) The exam is **90 minutes** long.

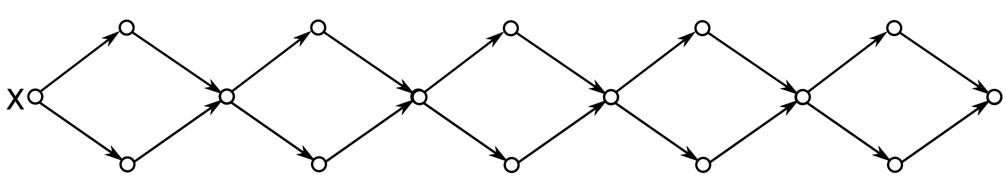
Problem 1. Give the adjacency *list* of the following unweighted directed graph.



Problem 2.

(a) Show two different DFS trees of the following graph rooted at x.

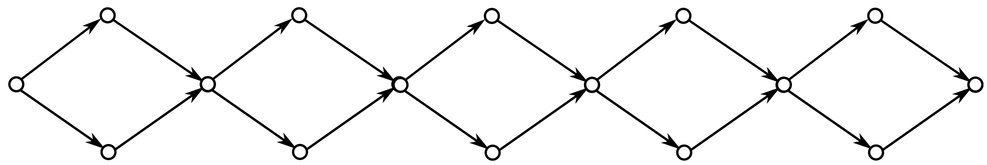
(b) What is the number of different DFS trees of this graph that are rooted at x?



Problem 3. A topological sort of a directed acyclic graph $G = (V, E)$ is an ordering of its vertices such that for each directed edge $(u, v) \in E$ the vertex u becomes before the vertex v.

(a) Give a topological sort of the the following graphs.

(b) What is the number of different topological sorts of this graph?

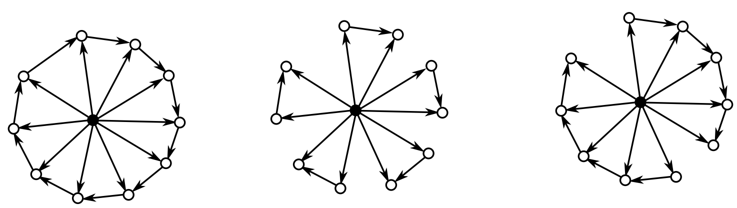


Problem 4. We know that Dijkstra's algorithm is correct if the edge weights of the input graph are all positive.

(a) Give an example of a graph with negative edges and a source vertex from which Dijkstra computes shortest paths correctly.

(b) Give an example of a graph with negative edges (but no negative cycle) and a source vertex from which Dijkstra does *not* compute the shortest paths correctly.

Problem 5. Suppose the following graphs are arbitrarily weighted with negative, positive, or zero edge weights, but there is no negative cycle. For each of them, what is the minimum number of required *phases* of Shimbel's (Bellman-Ford's) algorithm to compute all shortest paths from the solid vertex in the center. (We know that $V - 1$ phases is sufficient for any graph, can you give better bounds for any of these graphs?).



Problem 6. What is the best algorithm for computing single source shortest paths in each of the following classes of graphs. Give the running time of your suggested algorithms.

(a) DAGs with negative and positive edges

(b) General graphs with only positive edges.

(c) General graphs with positive and negative edges but no negative cycles.

Problem 7. Which of the following statements are true or false?

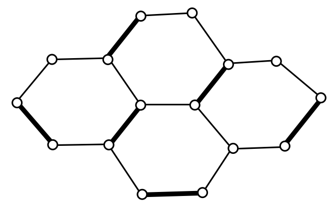
(a) Let G be a weighted graph, let C be a cycle in G, and let e be an edge in C. If e has the maximum weight among the edges of C then it is in no minimum spanning tree of G.

(b) Let G be a weighted graph with distinct edge weights, let C be a cycle in G, and let e be an edge in C. If e has the maximum weight among the edges of C then it is in no minimum spanning tree of G.

(c) Let G be a weighted graph, let (S, T) be a partition of the vertices of G, and let e be an edge of (S, T) (an edge with exactly one endpoint in S and exactly one endpoint in T). If e has the minimum weight among the edges of (S, T) then it is in every minimum spanning tree.

(d) Let G be a weighted graph with distinct edge weights, let (S, T) be a partition of the vertices of G, and let e be an edge of (S, T) (an edge with exactly one endpoint in S and exactly one endpoint in T). If e has the minimum weight among the edges of (S, T) then it is in every minimum spanning tree.

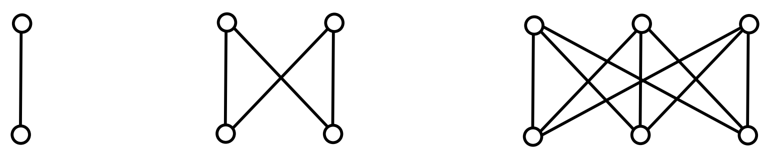
Problem 8. Answer the following questions about the matching in the figure.



(a) Show an alternating path that is *not* augmenting.

(b) Show two disjoint augmenting paths.

Problem 9. $K_{m,n} = (X \cup Y, E)$ denotes a complete bipartite graph, $|X| = n$, $|Y| = m$, and E contains all possible edge between X and Y. The figure illustrates $K_{1,1}$, $K_{2,2}$, and $K_{3,3}$.



(a) Give the Tutte matrix and its determinant for $K_{2,2}$.

(b) How many monomials does the Tutte determinant of $K_{n,n}$ have? (A monomial is a polynomial that is composed of only one term, and each polynomial is composed of a set of monomials. For example: $x_{1,1}x_{2,2}x_{3,3} + x_{1,2}x_{2,1}x_{3,3} + x_{1,3}x_{2,1}x_{3,2}$ is composed of three monomials.)

Problem 10. What is the minimum number of edges to add to each of the following graphs to make it Eulerian?

(a) A path P_n, with n vertices.

(b) A full binary tree (a rooted tree, in which each internal vertex has exactly two children) with n vertices.

(c) A complete graph K_n with n vertices, if n is even.

(d) A complete graph K_n with n vertices, if n is odd.

Problem 11. In the last homework, we tried (hard) to come up with an algorithm to compute the shortest paths between k partitions of vertices of a graph with negative, positive, and zero edge weights (but no negative cycles) in $O(V^2 + kE \log E)$ time. Show that if such an algorithm exists then there exists an algorithm to compute the shortest path between a single source s and a single target t in a graph with negative, positive and zero edge weights (but no negative cycles) in $O(V^2 + E \log E)$ time (Hint: assume such an algorithm exists, and use it as a black box to find the shortest path between s and t).