Lecture 6

Monday, November 7, 2022 2:43 PM

Independence E, F are independent if $P(E \cap F) = P(E)P(F)$.

Conditional prob.

$$\underbrace{P(E|F)}_{} \overset{\text{def}}{=} \frac{P(E \cap F)}{P(F)}$$

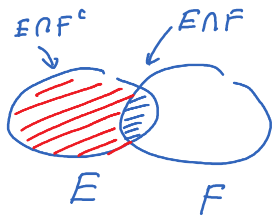
If E & F are ind then

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{P(E)\cancel{P(F)}}{\cancel{P(F)}} = P(E)$$

Law of total probability

$$P(E) = \underbrace{P(E|F)}_{\frac{P(E\cap F)}{\cancel{P(F)}}}\cancel{P(F)} + \underbrace{P(E|F^c)}_{\frac{P(E\cap F^c)}{\cancel{P(F^c)}}}\cancel{P(F^c)}$$

$$\underbrace{P(E \cap F) + P(E \cap F^c)}_{P(E)}$$



$$P(E) = P(E|F)P(F) + P(E|F^c)P(F^c)$$

Ex

Al & Bill democrat candidates

George incumbent (Republican)

$P(G|A) = \frac{1}{2}$ $P(A) = \frac{2}{3}$

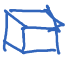
$P(G|B) = \frac{1}{3}$ $P(B) = \frac{1}{3}$

$P(G) = ?$

$A = B^c$

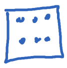
Law of total prob: $P(G) = P(G|A)P(A) + P(G|B)P(B) = \frac{1}{2} \times \frac{2}{3} + \frac{1}{3} \times \frac{1}{3} = \frac{4}{9}$

Ex

 roll: $\begin{cases} \boxed{\leq 4} \to \text{roll 1 more time} \to X \\ \boxed{\geq 5} \to \text{roll 2 more times} \to \text{sum (2 times)} \to X \end{cases}$

$P(X=5) = ?$

 $\to$  $X = 4$

 $\to$   $X = 3$

E = "$X = 5$" $P(E) = ?$

F = "≤ 4 the first roll"

F^c = "≥ 5 the first roll"

$$P(E) = \underbrace{P(E|F)}_{\frac{1}{6}}\underbrace{P(F)}_{\frac{4}{6}} + \underbrace{P(E|F^c)}_{\frac{1}{12}}\underbrace{P(F^c)}_{\frac{2}{6}} = \boxed{\frac{5}{36}}$$

$\downarrow$

$$P(E|F) = P(X=5 \mid \text{"}\leq 4 \text{ first roll"}) = P(\text{second roll gives } 5) = \frac{1}{6}$$

independent

$$P(E|F^c) = P(X=5 \mid \text{"}\geq 5 \text{ first roll"}) = P(\text{second} + \text{third} = 5)$$

$$= P(\{13, 22, 31\})$$

$$= \frac{3}{36} = \frac{1}{12}$$

Expectation

random number

- # hours of sleep
- face value in a die roll
- # customer of a store
- # emails you get / day
- award for lottery

average ← expectation

Ex X random number (random variable)

Suppose X can only take value 1 or 2.

$$P(X=1) = \frac{1}{3}$$

$$P(X=2) = \frac{2}{3}$$

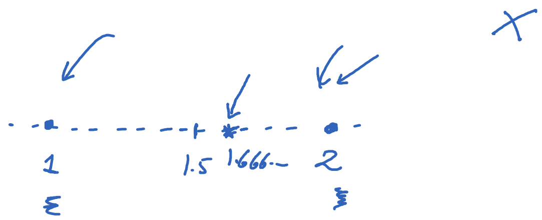
what is the average of X?

Is it 1.5?

Expectation of X, denoted by

$$E(X)$$

$$E(X) = \frac{5}{3}$$



On average, in every 3 times we examine the value of X, $X=1$ once, $X=2$ twice.

$$\frac{1+2+2}{3} = \frac{5}{3} \approx 1.666.$$

If X can only take values $a_1, a_2, a_3, \dots, a_n$,

Then $E(X) = a_1 P(X=a_1) + a_2 P(X=a_2) + \dots + a_n P(X=a_n)$.

Ex sleeping hours : $4, 5, 6, 7, 8, 9, 10 \leftarrow X$

$4 \to 5\%$, $5 \to 10\%$, $6 \to 15\%$, $7 \to 25\%$, $8 \to 30\%$, $9 \to 10\%$, $10 \to 5\%$

$$P(X=4) = \frac{5}{100}$$

$$P(X=5) = \frac{10}{100} \quad \dots$$

$$E(X) = 4\underbrace{P(X=4)}_{\frac{5}{100}} + 5\underbrace{P(X=5)}_{\frac{10}{100}} + 6\underbrace{P(X=6)}_{\frac{15}{100}} + 7P(X=7) + 8P(X=8) + 9P(X=9) + 10P(X=10)$$

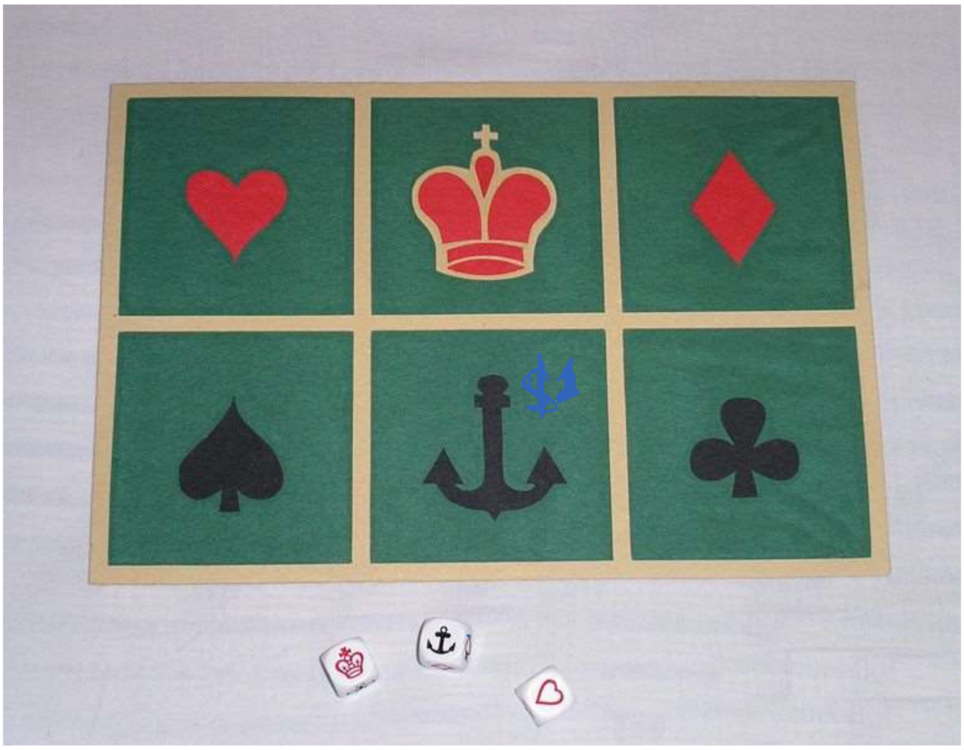
$$= \boxed{7.15}$$

Game of chance

chess, tic-tac-toe, card, ...

Bầu cua tôm cá $\longleftrightarrow$ Crown and anchor

Bầu → squash, cua → crab, tôm → shrimp, cá → fish



$X = \$$ you win

$X = 1, 2, 3, -1$

$EX = 0$ fair game

> 0

< 0

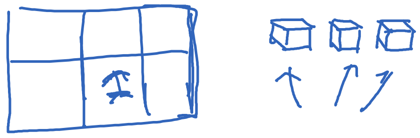
$X = 1, 2, 3, -1$

$E(X) = 1\,P(X=1) + 2\,P(X=2) + 3\,P(X=3) + (-1)\,P(X=-1)$

$P(X=1) = P(1 \text{ anchor in } 3 \text{ dice}) = P(1 \text{ success out of } 3 \text{ experiment})$

$= C(3,1)\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^2 = \frac{75}{216}$

$X = \$$ you win



3 Bernoulli experiments

chance of success $= \frac{1}{6}$

$P(X=2) = P(2 \text{ success out of } 3 \text{ exp}) = C(3,2)\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^1 = \frac{15}{216}$

$P(X=3) = P(3 \text{ '' '' ''}) = C(3,3)\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^0 = \frac{1}{216}$

$P(X=-1) = P(0 \text{ success out } 3 \text{ exp}) = C(3,0)\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^3 = \frac{125}{216}$

$E(X) = -\frac{17}{216} < 0$