Lecture 7

Tuesday, November 15, 2022 4:49 PM

Power $a^n = \underbrace{a \times a \times \cdots \times a}_{n \text{ times}}$

n · exponent

a : base

a^n: n'th power of a.

$$a^n a^m = \underbrace{a \times a \times \cdots \times a}_{n} \times \underbrace{a \times a \times \cdots \times a}_{m} = a^{n+m}$$

$$\frac{a^n}{a^m} = \frac{a \times a \times \cdots \times a}{a \times a \times \cdots \times a} = \frac{\overbrace{a \times a \times \cdots \times a}^{n-m}}{1} = a^{n-m}$$

n = m:

$$\underbrace{\frac{a^n}{a^n}}_{1} = a^{n-n} = a^0$$

$$\boxed{a^0 = 1}$$

$$\frac{1}{a^2} = \frac{a^0}{a^2} = a^{0-2} = a^{-2}$$

$$\frac{1}{a^4} = a^{-4}$$

$$(a^n)^m = \underbrace{a^n a^n \cdots a^n}_{m \text{ times}} = \underbrace{\underbrace{aa \cdots a}_{n} \; \underbrace{aa \cdots a}_{n} \cdots \underbrace{aa \cdots a}_{n}}_{m \text{ blocks}} = a^{nm}$$

$$(a^2)^3 = a^{2 \times 3} = a^6$$

$$(a^2)^3 \neq a^{2+3} = a^5$$

$$a^2 a^3 = a^{2+3} = a^5$$

$$(ab)^2 = abab = a^2 b^2$$

$$(ab)^n = a^n b^n$$

$$(abc)^2 = abc\,abc = a^2 b^2 c^2$$

$$\boxed{(abc)^n = a^n b^n c^n}$$

Ex

$$\frac{(xy^2)^2}{(x^2y)^3} = \frac{x^2(y^2)^2}{(x^2)^3y^3} = \frac{x^2y^4}{x^6y^3} = \frac{1}{x^4}\frac{y}{1} = \frac{y}{x^4} = yx^{-4}$$

$a^{\boxed{n}}$

$2^2 = 4$

$2^3 = 8$

$2^x = 6$, what is x? $\quad x = \log_2 6$ — logarithm of 6 to the base 2

$\boxed{\log_2}$

$3^x = 7 \quad x = \log_3 7$

$\boxed{\log_3}$

$\boxed{\log_4}$ $\boxed{\log_5}$ ...

Imagine: $\boxed{\log_{10}}$ is a button of the calculator

Find $\log_2 3$.

$x = \log_2 3 \rightsquigarrow 2^x = 3$

$2 = 10^{\log_{10} 2}$

$10^c = 2 \quad c = \log_{10} 2$

$$3 = \left(10^{\log_{10} 2}\right)^x = 10^{x \log_{10} 2}$$

$$x \log_{10} 2 = \log_{10} 3$$

$$\boxed{\log_2 3 = \frac{\log_{10} 3}{\log_{10} 2}}$$

Rule

$$\boxed{\log_a b = \frac{\log_{10} b}{\log_{10} a}}$$

$\log_{10}$ is called common logarithm → shortened as log

$\log 2$

Ex Find x such that $3^{\boxed{2x-1}} = 5$

$$2x-1 = \log_3 5 = \frac{\log 5}{\log 3}$$

$$2x = \frac{\log 5}{\log 3} + 1$$

$$x = \frac{\frac{\log 5}{\log 3} + 1}{2} \approx 1.2325$$

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interest / loan / rate / morgage / ...

Imagine: have \$1000 → bank

interest rate = 0.01% = 0.0001

After 1 year $\underbrace{1000}+\underbrace{1000 \times 0.0001} = 1000 + 0.1 = 1000.10$

After 2 years: $\underbrace{1000.10}_{\text{balance}} + 1000.10 \times 0.0001 = 1000.20001$

- Original amount : P
- Interest rate : r
- Compounding period.

Ex

Income : \$50,000 ← original amount

Raise rate. 4% = 0.04 ← interest rate

2nd year. 50 000 + 0.04 × 50000 = 52 000

3rd year : 52000 + 0.04 × 52000 ≈ 52 020

4th year. 52 020 + 0.04 × 52 020 ≈ - - - -

} compounding period = 1 year.

Ex bank → \$50,000

If compounding period = 1 quarter

Each quarter ↑ 1%

Q1 50000 + 50000 × 1%

Q2 . ↓ + ↓ × 1%

Q3 . . . .

P : original

r : int. rate per year

# of period is n

Originally: P dollars

period 1 : $P + P \cdot \frac{r}{n} = P\left(1 + \frac{r}{n}\right)$

period 2 : $P\left(1+\frac{r}{n}\right)1 + P\left(1+\frac{r}{n}\right)\frac{r}{n} = P\left(1+\frac{r}{n}\right)\left(1+\frac{r}{n}\right)$

$$= P\left(1+\frac{r}{n}\right)^2$$

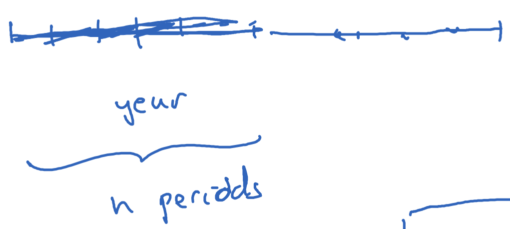
period 3 : $P\left(1+\frac{r}{n}\right)^3$

period k $P\left(1+\frac{r}{n}\right)^k$

$A(k)$: balance after k period

$$A(k) = P\left(1+\frac{r}{n}\right)^k$$

After t years : $k = nt$ periods



Balance after t years is $P\left(1+\frac{r}{n}\right)^{nt} \approx P(e^r)^t = Pe^{rt}$

$\left(1+\frac{r}{n}\right)^n \underset{n \text{ large}}{\approx} e^r$ Calculus I

$e \approx 2.7172\ldots$ Nepe number

Continuous compounding

after t year $\quad Pe^{rt}$

Ex $\left\{ \begin{array}{l} \$1000 \text{ in saving account} \\ r = 1\%/\text{year} \\ \text{compounding interest} \end{array} \right.$

when will we get \$1500 in saving account?

after t years $\quad \underbrace{P}_{1000} e^{rt} = 1000\, e^{0.01t} = 1500$

$$e^{0.01t} = \frac{1500}{1000} = 1.5$$

$$0.01t = \log_e 1.5 = \ln 1.5 \rightsquigarrow t = \frac{\ln 1.5}{0.01}$$

$\ln$ natural logarithm $\quad \boxed{t \approx 40.5465}$

$\log$. common log ..