Homework 8

1) (a) * Use long division:

$$\begin{array}{r|l} & x^2 - x - 2 \\ \hline x-1 & x^3 - 2x^2 - x + 1 \\ - & x^3 - x^2 \\ \hline & -x^2 - x \\ - & -x^2 + x \\ \hline & -2x + 1 \\ - & -2x + 2 \\ \hline & -1 \end{array}$$

Thus, $x^3 - 2x^2 - x + 1 = (x-1)\underbrace{(x^2-x-2)}_{\text{quotient}} \underbrace{-1}_{\text{remainder}}$

* Use synthetic division:

$$\begin{array}{c|cccc} 1 & 1 & -2 & -1 & 1 \\ & & 1 & -1 & -2 \\ \hline & 1 & -1 & -2 & -1 \end{array}$$

$\underbrace{1 \quad -1 \quad -2}_{\text{quotient is } x^2-x-2} \quad \underbrace{-1}_{\text{remainder}}$

(b) * Use long division:

$$\begin{array}{r|l} & 2x + 3 \\ \hline x^2 + 0x + 1 & 2x^3 + 3x^2 + x - 2 \\ - & 2x^3 + 0x^2 + 2x \\ \hline & 3x^2 - x - 2 \\ - & 3x^2 + 0x + 3 \\ \hline & -x - 5 \end{array}$$

Thus, $2x^3+3x^2+x-2 = (x^2+1)\underbrace{(2x+3)}_{\text{quotient}}\underbrace{-x-5}_{\text{remainder}}$

(c)

$$
\begin{array}{r|l}
 & x^2 - x \\
\hline
x^2+x+1 & x^4+0x^3+0x^2-x-1 \\
 & -\underline{(x^4+x^3+x^2)} \\
 & \quad -x^3-x^2-x \\
 & \quad -\underline{(-x^3-x^2-x)} \\
 & \qquad\qquad 0-1
\end{array}
$$

Thus,

$$x^4-x-1 = (x^2+x+1)\underbrace{(x^2-x)}_{\text{quotient}}\underbrace{-1}_{\text{remainder}}$$

(d) * Use long division:

$$
\begin{array}{r|l}
 & 2x^2+3x+3 \\
\hline
x-2 & 2x^3-x^2-3x-6 \\
 & -\underline{(2x^3-4x^2)} \\
 & \quad 3x^2-3x \\
 & \quad -\underline{(3x^2-6x)} \\
 & \qquad 3x-6 \\
 & \qquad -\underline{(3x-6)} \\
 & \qquad\quad 0
\end{array}
$$

Thus, $2x^2-x^2-3x-6 = (x-2)(2x^2+3x+3)$

* Use synthetic division:

$$\begin{array}{r|rrrr} 2 & 2 & -1 & -3 & -6 \\ & \downarrow & 4 & 6 & 6 \\ \hline & 2 & 3 & 3 & 0 \end{array}$$

quotient is $2x^2+3x+3$ remainder is 0

2) (a) $f(x) = x^3 - 2x^2 - 5x + 6$

Guess a root $x = p/q$ of $f(x)$: p is a divisor of 6 and q is a divisor of 1.

p/q could be $\pm 1, \pm 2, \pm 3, \pm 6$.

We can see that $f(1) = 1-2-5+6 = 0$, so $x=1$ is a root of f.
That means $x-1$ is a factor of $f(x)$. We use synthetic division to find the quotient:

$$\begin{array}{r|rrrr} 1 & 1 & -2 & -5 & 6 \\ & \downarrow & 1 & -1 & -6 \\ \hline & 1 & -1 & -6 & 0 \end{array}$$

$x^3-2x^2-5x+6 = (x-1)(x^2-x-6)$

We continue to factor x^2-x-6.
We see that -2 is a root, so $x-(-2) = x+2$ is a factor of x^2-x-6.

$$\begin{array}{r|rrr} -2 & 1 & -1 & -6 \\ & \downarrow & -2 & 6 \\ \hline & 1 & -3 & 0 \end{array}$$

$x^2-x-6 = (x+2)(x-3)$

Therefore, $x^3-2x^2-5x+6 = (x-1)(x+2)(x-3)$.
The roots are $1, -2, 3$.

(b) $f(x) = x^4 - 9x^2 - 4x + 12$

Guess a root of the form p/q where $q = 1$ and p is a divisor of 12.

By testing $x = \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$, we see that $x = 1$ is a root.

By the Factor theorem, $x-1$ is a factor of $f(x)$.

$$\begin{array}{r|rrrrr} 1 & 1 & 0 & -9 & -4 & 12 \\ & & 1 & 1 & -8 & -12 \\ \hline & 1 & 1 & -8 & -12 & 0 \end{array}$$

$f(x) = (x-1)\underbrace{(x^3 + x^2 - 8x - 12)}$

note that $x = -2$ is a root.

$$\begin{array}{r|rrrr} -2 & 1 & 1 & -8 & -12 \\ & & -2 & 2 & 12 \\ \hline & 1 & -1 & -6 & 0 \end{array}$$

$x^3 + x^2 - 8x - 12 = (x+2)\underbrace{(x^2 - x - 6)}$

$x = -2$ is a root

$$\begin{array}{r|rrr} -2 & 1 & -1 & -6 \\ & & -2 & 6 \\ \hline & 1 & -3 & 0 \end{array}$$

$x^2 - x - 6 = (x+2)(x-3)$

Therefore,

$$f(x) = (x-1)(x+2)(x+2)(x-3)$$

The roots are $1, -2, 3$.

(c) $f(x) = 12x^3 - 4x^2 - 3x + 1$

Guess roots $x = \pm 1, \pm \frac{1}{2}, \pm \frac{1}{3}, \pm \frac{1}{4}, \pm \frac{1}{6}, \pm \frac{1}{12}$.

We see that $x = \frac{1}{2}$ is a root.

Thus, $x - \frac{1}{2}$ is a factor. This also mean $2(x - \frac{1}{2}) = 2x - 1$ is a factor.

$$
\begin{array}{r|l}
 & \quad 6x^2 + x - 1 \\ \hline
2x-1 & 12x^3 - 4x^2 - 3x + 1 \\
 & -\underline{(12x^2 - 6x^2)} \\
 & \qquad 2x^2 - 3x \\
 & \quad -\underline{(2x^2 - x)} \\
 & \qquad\quad -2x + 1 \\
 & \qquad -\underline{(-2x + 1)} \\
 & \qquad\qquad\quad 0
\end{array}
$$

Thus, $f(x) = (2x-1)\underbrace{(6x^2+x-1)}_{x=-\frac{1}{2}\text{ is a root}}$

By Factor's thm, $x - \left(-\frac{1}{2}\right) = x + \frac{1}{2}$ is a factor of $6x^2+x-1$.

This also means $2\left(x+\frac{1}{2}\right) = 2x+1$ is a factor of $6x^2+x-1$.

$$
\begin{array}{r|l}
 & \quad 3x - 1 \\ \hline
2x+1 & 6x^2 + x - 1 \\
 & -\underline{(6x^2 + 3x)} \\
 & \qquad -2x - 1 \\
 & \quad -\underline{(-2x - 1)} \\
 & \qquad\qquad 0
\end{array}
$$

$6x^2+x-1 = (2x+1)(3x-1)$

Therefore, $f(x) = (2x-1)(2x+1)(3x-1)$.

The roots are $\frac{1}{2}, -\frac{1}{2}, \frac{1}{3}$.