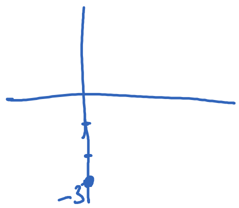
# Lecture 8

Friday, December 2, 2022    12:55 PM

Last time quadratic function

$f(x) = ax^2 + bx + c$ (general form)



$f(x) = x^2 - 2x - 3 = 0$

standard form

$f(x) = a\underbrace{(x-h)^2} + k$

$(x-h)^2 = x^2 - 2hx + h^2$

$x^2 - 2x - 3$

$x^2 - 2x - 3 = \underbrace{x^2 - 2x + 1}_{(x-1)^2} - 4$

$f(x) = (x-1)^2 - 4 = 0 \rightsquigarrow (x-1)^2 = 4$

$x - 1 = \pm 2$ < $x = 3$, $x = -1$

There is another method to find root of a quadratic function

$x^2 - 2x - 3 = (x+1)\boxed{(x-3)}$ factorization

$x\square + 1\square = x(x-3) + 1(x-3)$

$= x^2 \underbrace{-3x} + \underbrace{x} - 3$

$= x^2 - 2x - 3$

$(x+1)(x-3) = 0 \rightsquigarrow$ either $\underbrace{x+1=0}_{x=-1}$ or $\underbrace{x-3=0}_{x=3}$

Polynomials

A polynomial is a function of the form

nomial

$$f(x) = a_n \boxed{x^n} + a_{n-1} \boxed{x^{n-1}} + \cdots + a_1 \boxed{x^1} + a_0 \boxed{x^0}$$

$a_n, a_{n-1}, \ldots, a_1, a_0$ given

Ex

$$f(x) = x^2 - 3x + 4 = 1\,x^{\boxed{2}} - 3x + 4 = a_2 x^2 + a_1 x + a_0$$

$$= \cancel{0x^3} + 1\,x^2 - 3x + 4$$

$$= \cancel{0x^4} + \cancel{0x^3} + 1x^2 - 3x + 4$$

$$f(x) = a_n \boxed{x^n} + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \qquad (a_n \neq 0)$$

$n =$ degree of polynomial $f$

$a_n =$ highest coefficient

$a_0 =$ free coefficient

$a_n, a_{n-1}, \ldots, a_1, a_0$ coefficients.

Ex

$$f(x) = x^5 - x^2 - 1 = 1x^5 + 0x^4 + 0x^3 + (-1)x^2 + 0x + (-1)$$

fifth-degree polynomial

high coef

free coeff

poly. of degree 5

**Fact.** a polynomial of degree $n$ cannot have more than $n$ roots

x-intercept

$x$ is a root of $f$ if $f(x) = 0$.

Multiplication & division of polynomials

$f(x) = x^2+x+1$, degree $= 2$ quadratic polynomial

$g(x) = x+2$ , degree $= 1$

$$f(x)\,g(x) = (x^2+x+1)\boxed{(x+2)} = x^2\square + x\square + 1\square$$
$$= x^2(x+2) + x(x+2) + 1(x+2)$$
$$= x^3 + 2x^2 + x^2 + 2x + x + 2$$
$$= x^3 + 3x^2 + 3x + 2, \quad \text{degree } 3$$

cubic polynomial

## How to divide a poly. by a poly?

$4 \div 2 = 2$ with remainder 0.

$5 \div 2 = 2$ with remainder 1 $\qquad 5 = 2\times 2 + 1$

dividend = divisor × quotient + remainder

Note: remainder < divisor

$5 \div 2$

How to divide $f(x) = x^5 - x - 1$ by $g(x) = x^2 + x - 1$ ?

$$f(x) = g(x)\,q(x) + r(x)$$

dividend, divisor, quotient, remainder

degree $r$ < degree $g$

# Polynomial long division

$f(x) = x^5 - x - 1$

$g(x) = x^2 + x - 1$

quotient

$$\begin{array}{rl}
 & x^3 + x^2 + 1 \\
x^2 + x - 1 & \overline{\big)\, x^5 + 0x^4 + 0x^3 + 0x^2 - x - 1} \\
- & \underline{x^5 + x^4 - x^3} \\
 & \quad x^4 + x^3 + 0x^2 \\
- & \quad \underline{x^4 + x^3 - x^2} \\
 & \qquad\qquad x^2 - x + 0 \\
- & \qquad\qquad \underline{x^2 + x - 1} \\
 & \qquad\qquad 0 \quad -2x + 1 - 1
\end{array}$$

quotient $= x^3 + x^2 + 1$

remainder $= -2x$

$$x^5 - x - 1 = (x^2 + x - 1)(x^3 + x^2 + 1) - 2x$$

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Special case of division

[divisor $= x - c$]

Ex. $(x^3 + x^2 + x + 1) \div (x - 2)$

$15 = 15x^0$

$$\begin{array}{rl}
 & x^2 + 3x + 7 \\
x - 2 & \overline{\big)\, x^3 + x^2 + x + 1} \\
- & \underline{x^3 - 2x^2} \\
 & \quad 3x^2 + x \\
- & \quad \underline{3x^2 - 6x} \\
 & \qquad 7x + 1 \\
- & \qquad \underline{7x - 14} \\
 & \qquad\quad 15
\end{array}$$

$$x^3 + x^2 + x + 1 = (x - 2)(\underline{x^2 + 3x + 7}) + 15$$

$(x^3+x^2+x+1) \div (x-2)$

Synthetic division.

$$
\begin{array}{c|cccc}
2 & 1 & 1 & 1 & 1 \\
 & \downarrow & 2 & 6 & 14 \\
\hline
 & \underbrace{1 \quad 3 \quad 7}_{\text{quotient}} & & & \underset{\sim}{15} \\
\end{array}
$$

quotient $\qquad$ remainder

$$= 1x^2+3x+7$$

Ex $\quad (x^4+x^2+1) \div (x+1) \qquad x+1 = x-(\underline{-1})$

$$
\begin{array}{c|ccccc}
-1 & 1 & 0 & 1 & 0 & 1 \\
 & \downarrow & -1 & 1 & -2 & 2 \\
\hline
 & \underline{1} & -1 & 2 & -2 & \underset{\sim}{3} \\
\end{array}
$$

quotient $\qquad$ remainder

$$x^4+x^2+1 = \underbrace{(x+1)}_{\text{degree}=1}(x^3-x^2+2x-2) + \underbrace{3}_{\text{degree}=0}$$

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Find roots of a polynomial

$f(x) = 2x^3+x^2-5x+2$

free coeff $= 2 \qquad 1, 2, -1, -2$

highest coe $= 2 \qquad 1, 2$

<u>Rational root test</u>:

$$\boxed{\text{If } x = \frac{p}{q} \text{ is a root then } p \text{ is a divisor of the free coeff, } q \text{ is a divisor of the highest coeff.}}$$

$1, 2, -\frac{1}{2}, -1, -2, \frac{1}{2}$

$f(1) = 2(1)^3+1^2-5(1)+2$
$= 0$

1 is a root of $f$

Bezout's lemma

[ If $f(x)$ has root $x=c$ then $\underbrace{f(x) = (x-c)\,q(x) + \cancel{0}}_{\text{factorization of } f}$. ]

$f(x) = 2x^3 + x^2 - 5x + 2$ has root $x = 1$

divide $f$ by $(x-1)$

$$\begin{array}{c|cccc} 1 & 2 & 1 & -5 & 2 \\ & & 2 & 3 & -2 \\ \hline & 2 & 3 & -2 & 0 \end{array}$$

$2x^3 + x^2 - 5x + 2 = (x-1)(2x^2 + 3x - 2)$

$x = -2$ is a root of

$1, -1, 2, -2, \frac{1}{2}, -\frac{1}{2}$

$2x^2 + 3x - 2 = (x - (-2))\,q(x) = (x+2)\underbrace{q(x)}_{2x-1} = \boxed{(x+2)(2x-1)}$

$(2x^2 + 3x - 2) \div (x+2)$

$$\begin{array}{c|ccc} -2 & 2 & 3 & -2 \\ & & -4 & 2 \\ \hline & \underbrace{2 \quad -1}_{\text{quotient}} & & \underbrace{0}_{\text{remainder}} \end{array}$$

$$\boxed{2x^3 + x^2 - 5x + 2 = (x-1)(x+2)(2x-1)}$$

$x = 1$

$x = -2$

$x = \frac{1}{2}$