Lecture 24

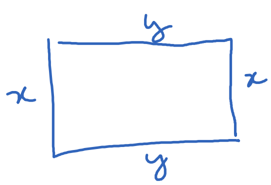
Thursday, November 17, 2022 8:29 AM

Questions ....

Application of derivatives to optimization problems

Optimization problem is the problem of find min/max of a function.

Let's consider the following problem a 10 ft string is bended to make a rectangle. What is the maximum of the enclosed area?

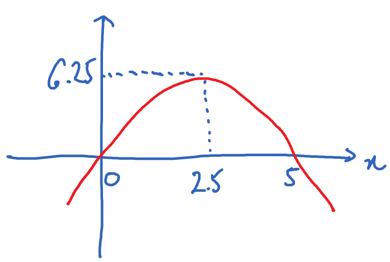


$$x+x+y+y=10$$

$$xy \longrightarrow \max$$

$$x+y=5 \rightsquigarrow xy = x(5-x) = \underbrace{-x^2+5x}_{f(x)}$$

Because f is a quadratic function, it is easy to find maximum by graphing



$$\max f = f(2.5) = 6.25$$

What if we are to deal with more complicated function where graphing can't be done by hand? Here is when derivative is very helpful.

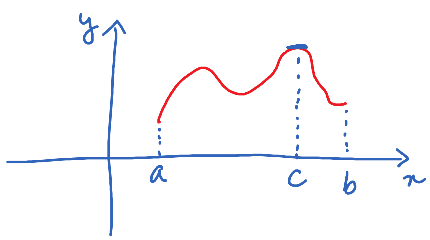
Fermat, a French mathematician, observed that if a function attains max or min on $[a,b]$ at a point c between two point a and b then c has to be either a "peak" or a "valley" and thus, $f'(c)=0$.

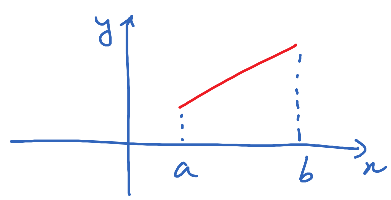
This turns the problem of finding min/max into a problem of solving the equation $f'(x)=0$. More specifically.

Fermat's lemma

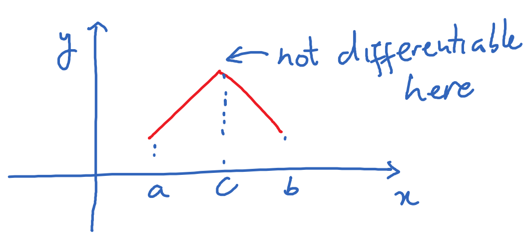
If f is differentiable on (a,b) and continuous on $[a,b]$ and has min or max at a point $c\in(a,b)$ then $f'(c)=0$.

Note that min/max can be attained at an endpoint (i.e. a or b).





Note that the condition that f is differentiable on (a,b) is needed. Without it, there may not be any $c\in(a,b)$ such that $f'(c)=0$.



Fermat's lemma gives us a procedure to find min/max.

To find $\min_{[a,b]} f$ or $\max_{[a,b]} f$, we do the following:

Step 1 find all the critical numbers of f on $[a,b]$.

a critical number is a number x such that $f'(x)=0$

Step 2 Compare the value of f at these critical numbers with $f(a)$ and $f(b)$.

The maximum of these values is the maximum of f on $[a,b]$.

" minimum " minimum "

Ex $f(x) = x^4 + x^3 + x^2 + 1$

Find $\max_{[-1,1]} f$ and $\min_{[-1,1]} f$.

Step 1 find the critical numbers of f

$$f'(x) = 4x^3 + 3x^2 + 2x = x(4x^2+3x+2) = 0$$

Either $x=0$ or $\underbrace{4x^2+3x+2=0}_{\text{doesn't have any real roots}}$.

Thus, $x=0$ is the only critical number of this function.

Step 2 Compare $f(0), f(-1), f(1)$

$$f(0) = 0^4 + 0^3 + 0^2 + 1 = 1$$

$$f(1) = 4$$

$$f(-1) = 2$$

Conclusion.

$$\max_{[-1,1]} f = 4, \text{ attained at } x = 1.$$

$$\min_{[-1,1]} f = 1, \text{ attained at } x = 0.$$

Work on the worksheet.