Lecture 27

Tuesday, November 29, 2022 8:35 AM

Questions....

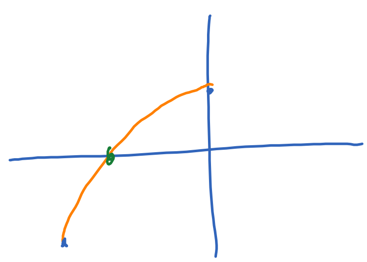
Work on worksheet 14:

① show that the polynomial $f(x) = x^3 + x^2 + 3x + 1$ has exactly one real root.

* Step 1: show that f has at least one root.
* Step 2: show that f has at most one root.

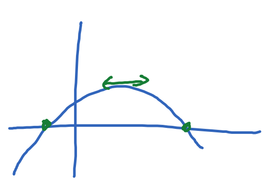
To do Step 1, we use the Intermediate Value theorem:

$$\left\{ \begin{array}{l} f(-2) < 0, \quad f(0) > 0 \\ f \text{ is continuous on } [-2,0] \end{array} \right. \rightsquigarrow f \text{ has a root between } -2 \text{ and } 0.$$



To do Step 2, we will use Rolle's thm.

Suppose by contradiction that f has another solution.



By Rolle's theorem, $f'(x) = 0$ for some value of x between the two roots of f.

$$f'(x) = 3x^2 + 2x + 3$$

This quadratic polynomial doesn't have any real root because

$$\Delta = 2^2 - 4(3)(3) < 0.$$

This is a contradiction!

(2) Show that the equation $x - \frac{1}{x^2} = 1$ has exactly one root.

$$f(x) = x - \frac{1}{x^2} - 1$$

Step 1: show that f has at least one root.

Step 2: show that f has at most one root.

* Be careful: $f(-1) < 0$, $f(2) > 0$, but we can't conclude that f has a root between -1 and 2. This is because the Intermediate value theorem requires f to be continuous on $[-1, 2]$, which is not the case.

Instead, we observe that any root of f has to be a positive number.

Indeed, for any $x < 0$, $f(x) = x - \frac{1}{x^2} - 1 < 0$.

Now we only need to look for roots of f on the interval $(0, \infty)$.

$f(1) < 0$, $f(2) > 0$ $\rightsquigarrow$ f has a root between 1 and 2.