Lab 1

In this lab, we will practice the following topics on Mathematica:

- Define piecewise functions and graph them
- Shade the region between curves
- Find an indefinite integral (antiderivative)
- Evaluate exactly or approximately a definite integral

1 Piecewise functions

A piecewise function is a function given by different formulas for each given interval. For example,

$$f(x) = \begin{cases} 2x & \text{if} \quad x \geq 0 \\ x^2 & \text{if} \quad x < 0 \end{cases}, \qquad g(x) = \begin{cases} \sin x & \text{if} \quad x < -1 \\ e^x & \text{if} \quad -1 \leq x \leq 1 \\ \cos x & \text{if} \quad x > 1 \end{cases}$$

are piecewise functions. We use the command **Piecewise** to define a piecewise function. The syntax is as follows:

Piecewise[{{val_1,$cond_1$},{val_2,$cond_2$},...}]

represents a piecewise function with values val_i in the regions defined by the conditions $cond_i$. If you want to define a function with default value val when none of the $cond_i$ apply, then use the following syntax:

Piecewise[{{val_1,$cond_1$},{val_2,$cond_2$},...},val]

(1) Use the following commands to define the function $f(x)$ above and plot it.

```
f[x_] := Piecewise[{{2 x, x >= 0}, {x^2, x < 0}}]
Plot[f[x], {x, -2, 2}]
```

Make sure that you don't miss the underscore after x. Press `Enter` to go from the first line to the second line. Then press `Shift+Enter` to execute the entire block.

(2) The first command is equivalent to

```
f[x_] := Piecewise[{{2 x, x >= 0}},x^2]
```

Try it out and plot the function $f(x)$ on the interval $[-2, 2]$.

(3) Define the function $g(x)$ above and plot it on the interval $[-2\pi, 2\pi]$. Do it in both ways as in the case of $f(x)$. Recall that π is typed in as Pi, and e^x is typed in as `Exp[x]` or `E^x`.

(4) Sometimes, we need to restrict the domain of a function. For example, let us consider a function $h(x) = \sin x$ if $1 \leq x \leq 8$ and undefined otherwise. That is to say

$$h(x) = \begin{cases} \sin x & \text{if} \quad 1 \leq x \leq 8 \\ \text{NaN} & \quad \text{otherwise} \end{cases}$$

The name NaN means "Not a Number" in Mathematica. Try the following commands:

```
h[x_] := Piecewise[{{Sin[x], 1 <= x <= 8}}, NaN]
h[0]
h[5]
```

What do you observe? Can you evaluate $h[5]$ numerically using the command **N[...]** ? (Review Lab 0 if you forget.)

(5) Plot the function $h(x)$ on the interval $[0, 10]$. What do you observe?

2 Shade the region between curves

It is useful to shade the region between curves on Mathematica, especially when you are asked to find its area. Instruction manuals and textbooks are full of pictures of this kind.

(6) Let us start simple. Given two functions, say $\sin x$ and $\cos x$, you can graph both of them on the same plot as follows.

```
Plot[{Sin[x],Cos[x]},{x,0,10}]
```

The list `{Sin[x],Cos[x]}` can be replaced by any list of functions `{func1,func2,...}` that you want to graph.

(7) You can label each curve by adding the option **PlotLegends** to the above command. Try the following:

```
Plot[{Sin[x],Cos[x]},{x,0,10}, PlotLegends->Automatic]
```

The curve labeled by 1 corresponds to the first function on the list (the sine function). The curve labeled by 2 corresponds to the second function on the list (the cosine function).

(8) Graph the following functions

$$\sin x,\ \cos x,\ \frac{1}{x^2+1},\ \ln x$$

together on the same plot on the interval $[1, 10]$. Label them using **PlotLegends**. Recall that $\ln x$ is typed in as `Log[x]`.

(9) To shade the region between two curves, we use the option **Filling**. Try the following:

```
Plot[{Sin[x], Cos[x]}, {x, 0, 10}, Filling -> {1 -> {2}}]
```

Notice that the region between curve 1 and curve 2 is shaded by the color of curve 1.

(10) You can shade the same region by the color of curve 2 by changing `1->{2}` to `2->{1}`. Try it.

(11) If you want to shade the region by a color of your choice, say Green, then change `1->{2}` to `1->{{2},Green}`. Try it. Then try `LightGreen` instead of `Green`.

(12) Now consider three curves $y = \sin x$, $y = \cos x$, and $y = x^2/3$. Graph them on the same plot on the interval $[0, \pi]$.

(13) You can see from the plot that there is a small region bounded by the three curves. We want to shade this region. The problem is that **Filling** can only shade the region between only two curves. Here is the trick. First, shade the region between curve 1 and curve 2.

```
p1 = Plot[{Sin[x], Cos[x], x^2/3}, {x, 0, Pi}, Filling -> {1 -> {2}},
   PlotLegends -> Automatic]
```

Then shade the region between curve 2 and curve 3 with white color.

```
p2 = Plot[{Sin[x], Cos[x], x^2/3}, {x, 0, Pi},
  Filling -> {2 -> {{3}, White}}, PlotLegends -> Automatic]
```

Then put p2 on top of p1 using the command **Overlay**.

```
Overlay[{p1, p2}]
```

(14) Graph four functions $\sin x$, $\cos x$, $4x - 2x^2$, $2 - x$ on the same plot on the interval $[0, 2]$.

(15) Shade the only region whose boundaries are made of all four curves. *Hint: in order to put p1 at the lowest layer, then p2 on top, then p3 on top, use the command* `Overlay[{p1,p2,p3}]`.

(16) Shade the region bounded by the curves $y = \sqrt{x}$, $x + y = 2$, and the x-axis. Recall that $\sqrt{x}$ is typed in as `Sqrt[x]`. *Hint: the x-axis has the equation $y = 0$.*

(17) Our next goal is to graph the parabola $y = x^2$ on the interval $[-3, 3]$ and shade the region under the parabola where $x \in [1, 2]$. Observe that the interval $[1, 2]$ on the x-axis is also the graph of the function

$$g(x) = \begin{cases} 0 & \text{if} \quad 1 \le x \le 2 \\ \text{NaN} & \quad\text{otherwise} \end{cases}$$

Therefore, the region to be shaded is the region between the graph of the function $g(x)$ and the function x^2.

```
g[x_] := Piecewise[{{0, 1 <= x <= 2}}, NaN]
Plot[{x^2, g[x]}, {x, -3, 3}, Filling -> {1 -> {2}}]
```

(18) Graph the sine curve $y = \sin x$ on the interval $[0, 2\pi]$ and shade the region under the curve where $x \in [0, \pi/2]$.

3 Find an antiderivative

Recall that $F(x)$ is an antiderivative of $f(x)$ if $F' = f$. To find an antiderivative of a function, we use the command **Integrate**. But be aware that not every function has an “expressible” antiderivative.

(19) For example, try the following to get an antiderivative of x^2.

```
Integrate[x^2,x]
```

(20) Define the function

$$f(x) = \begin{cases} 2xe^x & \text{if} \quad x \ge 0 \\ x^2 & \text{if} \quad x < 0 \end{cases}$$

using **Piecewise** and find an antiderivative. Call this function $F(x)$.

(21) Graph $f(x)$ and $F(x)$ on the interval $[-2, 2]$ on the same plot.

4 Evaluate definite integrals

To evaluate the exact value of a definite integral $\int_a^b f(x)dx$, we use the command **Integrate** with the syntax

```
Integrate[f[x],{x,a,b}]
```

Sometimes, it is impossible to get the exact value (some integrals are really tricky!) In that case, Mathematica may take a long time trying to compute. If you are using the Wolfram Cloud, you should terminate the execution by pressing the combination Alt+. if Mathematica takes longer than 30 seconds. Otherwise, you might soon run out of the precious 8 minutes quota of the month. If you are running Mathematica on your local computer, you don't have to worry about this issue.

If the command **Integrate** is taking too long to return a value, terminate it and try the command **NIntegrate** instead. It will instantly give you an approximate numerical value of the definite integral.

(22) Find the exact value of the integral $\int_1^4 \frac{1}{x}dx$.

(23) Try to evaluate the exact value of the integral $\int_0^2 e^{x^2}dx$ using **Integrate**. Does it give you a value?

(24) Now try **NIntegrate** instead of **Integrate**:

```
NIntegrate[E^(x^2),{x,0,2}]
NIntegrate[E^(x^2),{x,0,2},WorkingPrecision->8]
```

(25) Find the exact value of $\int_0^{\pi^2} \cos(\sqrt{x})dx$.

(26) Approximate the integral $\int_0^{\pi^2} \cos(\sqrt{x} - x)dx$ up to 10 digits after the decimal point.

5 To turn in

Submit your implementation of Exercises (1) - (26) as a single pdf file.