Lab 2

In this lab, we will practice the following topics on Mathematica:

- Factor and expand a polynomial
- Find the quotient and remainder of a polynomial division.
- Find partial fraction decomposition
- Approximate definite integrals using Riemann sums

1 Factor and expand a polynomial

(1) Type

```
Expand[(1+x)^2]
Expand[(1+x)^3]
Expand[(a+2b)^4]
```

(press `Enter` to go to from one line to the next) then press `Shift+Enter` to execute the block. What does the command `Expand` do?

(2) Expand the polynomial $(1+x)^5(2-x)^4$.

(3) Now try the command

```
TraditionalForm[Expand[(1+x)^5*(2-x)^4]]
```

What difference do you see in the output compared to the output of the previous command?

(4) The syntax `func[expr]` is equivalent to `expr // func`. The second way does not need the square brackets and is sometimes more convenient than the first way. Try the following.

```
(1+x)^5*(2-x)^4 // Expand // TraditionalForm
```

(5) Expand the polynomial $f(x) = (2x^2+x+1)^7(x-1)^3$ and arrange the terms in descending powers. What is the degree of f ?

(6) Try the commands

```
Factor[x^3+x^2-2]
x^3+x^2-2//Factor
x^3+x^2-2//Factor//TraditionalForm
```

What does the command `Factor` do?

(7) Factor the following polynomial and find all the real roots together with their multiplicities

$$f(x) = x^7 - 6x^6 + 11x^5 - 22x^3 + 20x^2 + 8x - 16$$

(8) To simplify the rational function $\dfrac{x^3+x^2-2}{x^2-3x+2}$, try the following commands

```
Simplify[(x^3+x^2-2)/(x^2-3x+2)]
Simplify[(x^3+x^2-2)/(x^2-3x+2)] // TraditionalForm
```

(9) Simplify the function $\cos^4(3x) - \sin^4(3x)$ and $\cos^4(3x) + \sin^4(3x)$. Note: the function $\cos^4(3x)$ is typed in as `Cos[3x]^4`.

2 Find the quotient and remainder of a polynomial division

Let $f(x)$ and $g(x)$ be two polynomials. The rational function $\frac{f(x)}{g(x)}$ can be written as

$$\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)}$$

where $q(x)$ is the quotient and $r(x)$ is the remainder whose degree is less than the degree of g. Normally, you find $q(x)$ and $r(x)$ by hand using long division. On Mathematica, you can find $q(x)$ and $r(x)$ via the command **PolynomialQuotientRemainder**. The syntax is

```
PolynomialQuotientRemainder[f(x),g(x),x]
```

The output is `{q(x),r(x)}`.

(10) Let us consider the division $(x^4 + 2x + 1) \div (x^2 + 1)$. Try the following:

```
PolynomialQuotientRemainder[x^4 + 2x + 1, x^2 + 1, x]
PolynomialQuotientRemainder[x^4 + 2x + 1, x^2 + 1, x]//TraditionalForm
```

What are the dividend, divisor, quotient, and remainder of this division?

(11) Find the quotient and remainder of the division $(x^6 - 1) \div (x - 2)$. Then use this result to find the integral

$$\int \frac{x^6 - 1}{x - 2} dx$$

without using the command **Integrate**. Please type out your answer.

3 Partial fraction decomposition

The command `Apart` decomposes a rational function into simple fractions.

(12) For example, the decompose the function

$$f(x) = \frac{x^4 - 1}{(x - 2)(x - 3)}$$

into partial fractions, try the following:

```
Apart[(x^4-1)/((x-2)*(x-3))]
Apart[(x^4-1)/((x-2)*(x-3))]//TraditionalForm
```

(13) Decompose the function

$$\frac{x^4 + 1}{x^5 + 4x^3}$$

into partial fractions.

(14) Decompose the function

$$\frac{1}{(x^2 - 9)^2}$$

into partial fractions.

4 Compute sums

(15) To compute $1+2+3+...+100$, we write this sum in *sigma notation* as $\sum_{k=1}^{100} k$. We can evaluate this formula with the command:

```
Sum[k,{k,1,100}]
```

(16) To compute $2^2 - 3^2 + 4^2 - 5^2 + ... - 99^2 + 100^2$, we write this sum in sigma notation as $\sum_{k=2}^{100} (-1)^{k+1} k^2$. We can evaluate this formula with the command:

```
Sum[(-1)^(k+1)*k^2,{k,2,100}]
```

(17) Use the command **Sum** to evaluate $\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + ... + \frac{1}{201}$. Then replace **Sum** by **NSum** to get the result as a decimal number instead of a fraction.
Hint: every odd number is of the form $2k+1$ *where* k *is a whole number.*

(18) Use the commands **Sum** and **NSum** to evaluate $\frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + ... - \frac{1}{201}$.

5 Approximate definite integrals using Riemann sums

To evaluate the definite integral $\int_a^b f(x)dx$, it is a common practice to find an antiderivative of $f(x)$ by using substitution and/or integration by parts. However, not all functions have an antiderivative of elementary form. In such a case, we can still find approximate values of the definite integral using Riemann sums.

We do so by dividing the interval $[a, b]$ into n equal subintervals of length $\Delta x = \frac{b-a}{n}$. The grid-points are $x_0 = a$, $x_1 = a + \Delta x$, $x_2 = a + 2\Delta x$,..., $x_n = a + n\Delta x = b$. In general, $x_k = a + k\Delta x$ for any k between 0 and n. As there are different ways to determine the height of the rectangular slats, there are different types of Riemann sums:

- Left endpoint Riemann sum:

$$L_n = \sum_{k=1}^{n} f(x_{k-1})\Delta x$$

- Right endpoint Riemann sum:

$$R_n = \sum_{k=1}^{n} f(x_k)\Delta x$$

- Midpoint Riemann sum:

$$M_n = \sum_{k=1}^{n} f\left(\frac{x_{k-1} + x_k}{2}\right)\Delta x$$

- Trapezoid Riemann sum:

$$T_n = \sum_{k=1}^{n} \frac{f(x_{k-1}) + f(x_k)}{2}\Delta x$$

(19) Knowing that the exact value of the integral

$$\int_1^2 x^2 dx$$

is $\frac{7}{3} = 2.333...$, it would be interesting to see how well these sums approximate this integral. We start by defining the function:

```
f[x_]:=x^2
a = 1
b = 2
```

Next, we will create a function called **Lsum** which receives the values of a, b, n and returns the value the left endpoint Riemann sum L_n. In the formula of L_n mentioned above, keep in mind that $x_{k-1} = a + (k-1)\Delta x$ and $\Delta x = (b-a)/n$.

```
Lsum[n_] :=  NSum[f[a + (k - 1)*(b - a)/n]*(b - a)/n, {k, 1, n}]
```

To find L_{10}, we use the command

```
Lsum[10]
```

(20) Find L_{100}, L_{1000}, L_{10000}. To see more decimal digits in the answers, for example 8 digits, add the option `WorkingPrecision->8` after `{k,1,n}` in the above command. What do you observe when increasing the value of n?

(21) Based on the function **Lsum** above, write the functions to compute the right endpoint, midpoint, and trapezoid Riemann sums. Name them **Rsum, Msum, Tsum**.

(22) To compare how well the sums **Lsum, Rsum, Msum, Tsum** approximate the integral

$$\int_1^2 x^2 dx$$

we need to draw a table. This table has 5 columns. The first column is the values of n. The second column is the values of **Lsum**. The third column is the values of **Rsum**, and so on. As far as rows are concerned, the first row is when $n = 100$. After each row, the value of n is increased by 200. The last row is when $n = 2100$.

```
Table[{n, Lsum[n], Rsum[n], Msum[n], Tsum[n]}, {n, 100, 2100, 200}] // TableForm
```

Which of those four Riemann sums yields the best result? In other words, as n increases, which column has the fastest convergence to the exact value 2.3333...?

(23) Repeat Problem 22 for the integral

$$\int_0^\pi \cos(x^2) dx.$$

You can use different starting, ending, spacing values of n. This is the case where one cannot find an elementary antiderivative. You can use the value computed by **NIntegrate**, a built-in function of Mathematica, as the exact value. If you forget how to use it, review Lab 1.

(24) To get a precision of 0.00001, how large must n be for each type of Riemann sums?

(25) To earn 1 point as extra credit, put the heading on each column in Problem 22.
Hint: search Google for the phrase "put heading for table in mathematica".

6 To turn in

Submit your implementation of Exercises (1) - (25) as a single pdf file.