Lab 3

In this lab, we will practice the following topics on Mathematica:

- Find integrals that have infinite bounds (improper integrals)
- Find integrals that have undetermined parameters
- Sketch solids of revolution, compute volume and surface area
- Visualize motions

1 Improper integrals

(1) In the command **Integrate**, you specify the integral bounds $\int_a^b$ by writing `{x,a,b}`. One or both of these bounds might be $\pm\infty$. Try the following:

```
Integrate[1/(x^2+1),{x,0,Infinity}]
NIntegrate[E^(-x^2),{x,-Infinity,0}, WorkingPrecision -> 20]
```

(press `Enter` to go to from one line to the next) then press `Shift+Enter` to execute the block.

For Problems 2-5, do the following:

- Plot the integrand.
- Shade the area which the integral represents.
- Use **Integrate** to get the exact value of the integral. Mathematica might fail to give you an answer.
- Use **NIntegrate** to get an approximate value. Round up to 8 digits after the decimal point.
- Explain why the integral is an improper integral.

(2) $\int_{-\infty}^{\infty} \frac{1}{x^2+1} dx$

(3) $\int_{-\infty}^{1} e^x \sin x dx$

(4) $\int_0^1 \ln(x) dx$

(5) $\int_0^1 \sin\left(\frac{1}{x}\right) dx$

2 Integrals with unspecified parameters

Sometimes, the integrand may contain a parameter whose value is unspecified. Consider two following examples,

$$\int_0^{a/2} \frac{1}{x^2 - a^2} dx, \qquad \int_0^{\infty} \frac{1}{x^2 + a^2} dx$$

(6) Find the first integral using **Integrate**.

(7) Find the second integral using **Integrate**. You will see that, unlike the first integral, the result depends on whether $a > 0$, $a < 0$, or $a = 0$.

To specify an assumption on the parameter a, we add an option **Assumptions** inside the **Integrate** command. For example, to specify that $a > 0$, we write

```
Assumptions->{a>0}
```

To specify that a is a natural number, i.e. 1,2,3,4,..., we write

```
Assumptions->{a \[Element] PositiveIntegers}
```

To specify that $a > 0$ and n is a natural number, we write

```
Assumptions->{a>0, n \[Element] PositiveIntegers}
```

(8) Try the command

```
Integrate[1/(x^2 + a^2), {x, 0, Infinity}, Assumptions -> a > 0]
```

(9) In the above command, change the assumption to $a < 0$ and then $a = 0$ (double equal sign). What do you observe?

(10) Find the integral

$$\int_0^\infty (a^2x^2 + b^2)^n \, dx$$

where $a < 0$, $b > 0$, and n is a negative integer.

3 Solid of revolution

Imagine a curve $y = f(x)$, $x \in [a, b]$. Rotate this curve about the x-axis. On this surface, all points that have the same x coordinate will also have the same distance toward the x-axis. This distance is $f(x)$. To sketch the surface, we use the syntax:

```
RevolutionPlot3D[f[x],{x,a,b},RevolutionAxis->{1,0}]
```

If the curve is rotated about the y-axis instead, we use the syntax:

```
RevolutionPlot3D[f[x],{x,a,b},RevolutionAxis->{0,1}]
```

In general, if the curve is rotated about the line $y = \frac{b}{a}x$ then we use the syntax:

```
RevolutionPlot3D[f[x],{x,a,b},RevolutionAxis->{a,b}]
```

(11) Sketch the surface obtained by rotation the parabola $y = x + \sin x$, $0 \leq x \leq 15$, about the x-axis. You can play with the surface using your mouse.

(12) Find the volume and surface area of this object. You might want to review your notes for those formula. Try to find the exact value first (using **Integrate**) and then an approximate value.

(13) Sketch the rotation of the curve $y = x^2$, $0 \leq x \leq 1$, about the x-axis and about the y-axis on the same plot.

For each number $a > 0$, we have a curve $y = x^a e^{-x}$, $0 \leq x \leq 20$. Each curve gives us a surface of revolution. Imagine a sequence of those surfaces is shown in order, one at a time, as a runs from 0 to 20. What we have is a mini movie (a sequence a frames)! Follow the steps below to see this 'movie'.

(14) First, for each a, define a frame

```
p[a_] := RevolutionPlot3D[x^a * E^(-2 x), {x, 0, 20},
            PlotRange -> {{-20, 20}, {-20, 20}}]
```

Next, put the frames together

```
Manipulate[p[a], {a, 0, 25, 0.25}]
```

Here 0 and 25 represent the starting and ending values of a, and 0.25 represents the increment step of a as it moves from the starting value to the ending value. Of course, you can change these values to what you prefer. Now look for the controller bar. It is in the plus-sign on Mathematica Notebook. If you use JupyterLab, the motion is already going.

(15) Repeat Problem 14 now with the curves $y = f(x)^a$ where

$$f(x) = 1 - |x|, \quad -1 \le x \le 1$$

Run the motion as a increases from 1 to a very large number. What do you observe?

(16) Repeat Problem 15, but now run the motion as a decreases from 1 down to 0. In the **Manipulate** command, you can put something like `{a,1,0,-0.01}`. What do you observe?

4 To turn in

Submit your implementation of Exercises (1) - (16) as a single pdf file.