Lab 6

In this lab, you will practice with sequences, series, and power series on Mathematica. Specifically, you will learn how to

- Define and evaluate terms and limit of a sequence,
- Evaluate partial sums and series,
- Find the radius of convergence and interval of convergence of a power series,
- Approximate a function by polynomials.

1 Define and evaluate terms and limit of a sequence

(1) Recall that a sequence is a function from $\mathbb{N}$ to $\mathbb{R}$. Thus, it can be defined the same way we define a function. For example, the sequence $a_n = \frac{(-1)^{n+1}}{2^n}$ can be defined as follows.

```
a[n_]:=(-1)^(n+1)/2^n
```

(2) To view the first 20 terms of this sequence, try the command

```
Table[{n, a[n]}, {n, 1, 20}] // TableForm
```

To convert the fractions into decimal-point numbers, replace `a[n]` in the command by `N[a[n]]`. If you want 4 significant digits, then use `N[a[n],4]` instead.

(3) Because a sequence is also a function, you can graph a sequence as you can with a function. However, since the variable n is a natural number, you will use `DiscretePlot` instead of `Plot`.

```
DiscretePlot[a[n], {n,1,20}, PlotRange -> Full]
```

The horizontal axis shows the indices n and the vertical axis shows the values of a_n.

(4) A sequence can also be defined by a recursive formula. For example, the Fibonacci sequence $a_1 = 1$, $a_2 = 1$, $a_n = a_{n-1} + a_{n-2}$ can be defined in Mathematica as

```
a[1] := 1
a[2] := 1
a[n_] := a[n - 1] + a[n - 2]
```

Find the first 20 terms of this sequence.

(5) Find the first 20 terms of the sequence defined recursively as

$$a_0 = a_1 = a_2 = 1, \quad a_n = a_{n-1} - 2a_{n-2} + a_{n-3}$$

Graph the sequence using `DiscretePlot`. Does the sequence appear to converge or diverge?

(6) If the general formula for a_n is available, you can simply use the command `Limit`. For example, to compute

$$\lim_{n\to\infty} \frac{2n^2 + n\sin(n)}{n^2 + 1}$$

try the following:

```
a[n_] := (2 n^2 + n*Sin[n])/(n^2 + 1)
Limit[a[n], n -> Infinity]
```

(7) Graph the above sequence using `DiscretePlot` to confirm the result.

The command `Limit` used above doesn't work well with sequences that are defined recursively. To find the limit of such a sequence, we use the command `RSolveValue` instead. The syntax is as follows:

```
RSolveValue[eqn, expr, n]
```

where `eqn` is a list consisting of the recursive formula and all initial conditions, `expr` is whatever expression you want to compute, and `n` is the index variable.

(8) For example, we want to find the limit of the sequence

$$a_{n+1} = \frac{a_n + na_{n-1}}{n+1}, \quad a_0 = 0, \ a_1 = 1.$$

Symbolically, the limit of a_n may be viewed as a_∞ (as if n is substituted by ∞). Thus, a_∞ is the value we want to find. Try the following:

```
RSolveValue[{a[n + 1] == (a[n] + n a[n - 1])/(n + 1),
            a[0] == 0, a[1] == 1}, a[Infinity], n]
```

(9) Consider a sequence defined recursively as follows:

$$a_{n+1} = \frac{2a_n + 3}{a_n + 4}, \quad a_0 = -2.$$

Find the limit of the sequence using `RSolveValue`. Then graph the sequence to confirm the result.

(10) Consider the series

$$\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$$

To define the k'th partial sum, do the following:

```
a[n_] := Log[n]/n^2;
s[k_] := Sum[a[n], {n, 1, k}];
```

(11) To get an idea what if the partial sums converge, find the first 50 (or more) partial sums using the command `Table` and graph them using `DiscretePlot`. From the table and graph, can you guess if the series $\sum a_n$ converges or diverges?

(12) You can find the value of the series by

```
Sum[a[n], {n,1,Infinity}]
```

If this command takes too long, change `Sum` to `NSum`.

(13) Repeat Exercises (10), (11), (12) for the series

$$\sum_{n=1}^{\infty} \frac{n^2 - 3n\sin n}{n^4 + 1}$$

2 Find radius and interval of convergence

Recall that a *power series centered at* x_0 is a series of the form $\sum a_n(x - x_0)^n$. On its interval of convergence, a power series defines a function. Not counting the endpoints, the interval of convergence is always a symmetric interval about the center of the power series. Consider the power series

$$\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$$

To find the radius of convergence, one can use either the Ratio test or the Root test. Let

$$a_n = \frac{(-3)^n x^n}{\sqrt{n+1}}$$

The Ratio test says that if the limit

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$$

exists and $L < 1$ then the series converges. If $L > 1$ then the series diverges. If $L = 1$ then the test fails.

(14) We compute the limit L as follows.

```
a[n_] := (-3)^n*x^n/Sqrt[n + 1]
Clear[n]
L = Limit[Abs[a[n + 1]/a[n]], n -> Infinity]
```

Here, `Abs` is the absolute value function.

(15) The values of x that makes $L < 1$ belongs to the interval of convergence. We solve the inequality $L < 1$ as follows.

```
Reduce[L < 1, x, Reals]
```

The option `Reals` in the above command is to indicate that we are interested in x as a real number (as opposed to complex number).

(16) You will see that the inequality $L < 1$ gives $x \in \left(-\frac{1}{3}, \frac{1}{3}\right)$. The radius of convergence is a half of the length of this interval, which is $R = \frac{1}{3}$. The endpoints $-1/3$ and $1/3$ have to be considered manually and separately. Mathematica can provide some insights as follows. Let $f(x)$ denote the value of the power series.

```
f[x_] := Sum[(-3)^n*x^n/Sqrt[n + 1], {n, 0, Infinity}]
```

We can attempt to evaluate f at $x = 1/3$ and $x = -1/3$.

```
f[1/3]
f[-1/3]
```

Mathematica will show a warning on the second command, indicating that f is not defined at $x = -1/3$. Therefore, the interval of convergence is $(-1/3, 1/3]$.

Now consider the power series

$$\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$$

Follow Exercises (14), (15), (16) to do the following exercises.

(17) What is the center of this power series? Find the radius of convergence.

(18) Find the interval of convergence.

(19) Approximate the value of the power series at $x = 0$.

3 Approximate the value of the power series

In many applications, it is helpful to approximate a function with polynomials. Polynomials are easier to take derivative or integral, and more computer-friendly because they involve only the addition, subtraction, and multiplication. To approximate a function by a polynomial, we simply truncate the Taylor series of the function. Recall that the Taylor series of a function f at x_0 is given by

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n, \quad \text{where} \quad a_n = \frac{f^{(n)}(x_0)}{n!}$$

Truncating this series at a power m, we get an m'th degree Taylor polynomial

$$T_m(x) = \sum_{n=0}^{m} a_n (x - x_0)^n$$

and f is approximated by $f(x) \approx T_m(x)$. The approximation is good when x is close to x_0, which is the center of the power series, and gets worse as x is far away from the center. To maintain a good approximation when x is far away from x_0, you will have to increase the degree m. The larger m is, the farther away x can be from x_0 and the approximation is still good.

To obtain the Taylor polynomials T_m centered at $x_0 = 0$ of a function f, we use the command **Series** with the syntax

```
Series[f, {x, 0, m}]
```

(20) For example, consider the function $f(x) = \sin x + \cos(x/\sqrt{2})$. Try the command

```
f[x_] := Sin[x] + Cos[x/Sqrt[2]]
Series[f[x], {x, 0, 7}]
```

The output will look something like

$$1 + x - \frac{x^2}{4} - \frac{x^3}{6} + \frac{x^4}{96} + \frac{x^5}{120} - \frac{x^6}{5760} - \frac{x^7}{5040} + O\left(x^8\right)$$

Therefore, the first seven Taylor polynomials $T_1, T_2, ..., T_7$ are

$$\begin{aligned}
T_1(x) &= 1 + x \\
T_2(x) &= 1 + x - \frac{x^2}{4} \\
T_3(x) &= 1 + x - \frac{x^2}{4} - \frac{x^3}{6} \\
&\cdots \\
T_7(x) &= 1 + x - \frac{x^2}{4} - \frac{x^3}{6} + \frac{x^4}{96} + \frac{x^5}{120} - \frac{x^6}{5760} - \frac{x^7}{5040}
\end{aligned}$$

The term $O(x^8)$ denotes the error term, which is the difference between the function f and the polynomial T_7.

(21) To see how well each Taylor polynomial approximates the function f, we graph them together on the same plot. For example, try the following to graph f and T_1 on the same plot.

```
T1[x_] := 1+x
Plot[{f[x], T1[x]}, {x,-3,3}]
```

(22) Use the fashion above to graph each of the functions $T_2, T_3, ..., T_7$ (one by one, not all at once) together with f on the same plot. What do you observe?

(23) One way to quantify how good the approximation $f(x) \approx T_m(x)$ is on the interval $x \in [-3,3]$ is by looking at the maximum value of $|f(x) - T_m(x)|$ on the interval $[-3,3]$. Try

```
MaxValue[{Abs[f[x] - T1[x]], -3 <= x <= 3}, x]
NMaxValue[{Abs[f[x] - T1[x]], -3 <= x <= 3}, x]
```

(24) For $m = 2, 3, ..., 7$, find the maximum of $|f(x) - T_m(x)|$ on the interval $[-3,3]$.

(25) How large does m have to be so that $f(x) \approx T_m(x)$ with an error less than 0.01 for any $x \in [-3,3]$?

4 To turn in

Submit your implementation of Exercises (1) - (25) as a single pdf file.