# Lecture 20

Wednesday, September 27, 2023 2:39 AM

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$\int_a^b f(x)\,dx$ is an <u>improper integral</u> if

- $f$ is unbounded on $[a,b]$, or
- $a = \pm\infty$ or $b = \pm\infty$

How to find improper integral?

If $f$ blows up at $b$, but is "nice" away from $b$ then

$$\int_a^b f(x)\,dx = \lim_{c\to b} \int_a^c f(x)\,dx$$

If one of the bounds is infinity then

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx.$$

If the limit doesn't exist, we say that the improper integral diverges.

Otherwise, it converges.

Ex

$$\int_{-1}^{8} \frac{1}{\sqrt[3]{x}}\,dx = \int_{-1}^{8} x^{-1/3}\,dx = \lim_{\varepsilon\to 0}\left(\int_{-1}^{-\varepsilon} x^{-1/3}\,dx + \int_{\varepsilon}^{8} x^{-1/3}\,dx\right)$$

$$= \lim_{\varepsilon\to 0}\left(\left.\frac{x^{2/3}}{2/3}\right|_{-1}^{-\varepsilon} + \left.\frac{x^{2/3}}{2/3}\right|_{\varepsilon}^{8}\right)$$

$$= \lim_{\varepsilon \to 0} \left( \underbrace{\frac{(-\varepsilon)^{2/3}}{2/3}}_{\to 0} + \frac{1}{2/3} - \frac{8^{2/3}}{2/3} - \underbrace{\frac{\varepsilon^{2/3}}{2/3}}_{\to 0} \right)$$

$$= \frac{1}{2/3} - \frac{8^{2/3}}{2/3} = \frac{3}{2} - \frac{3}{2} \cdot 4 = -\frac{9}{2}.$$

<u>Consequence</u>:

$$\int_0^1 x^p \, dx = \begin{cases} \infty & \text{if } p \leq -1 \\ \text{finite} & \text{if } -1 < p \end{cases}$$

$$\int_1^\infty x^p \, dx = \begin{cases} \text{finite} & \text{if } p < -1 \\ \infty & \text{if } p \geq -1 \end{cases}$$

<u>Comparison principle</u>:

If $|f(x)| \leq g(x)$ and $\int_a^b g(x)\,dx$ converges then $\int_a^b f(x)\,dx$ also converges.

<u>Ex</u>

$$\int_2^\infty \frac{1}{x^2-3}\,dx, \quad \int_0^\infty e^{-\sqrt{x}}\,dx, \quad \int_{-1}^1 \sin\left(\frac{1}{x}\right) dx$$