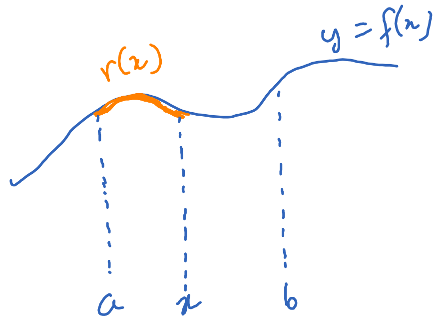
Lecture 21

Thursday, September 28, 2023 1:43 AM

* Prayer

We are now in Chapter 8. We learned derivative and integrals rather separately. Here we will see that they appear very naturally together in many applications.

Arclength: with integral, we know how to find the area between curves and volume of a solid. But what about the length of a curve? (To find the surface area of a solid, please wait until Math 214 - Multivariable Calculus.)

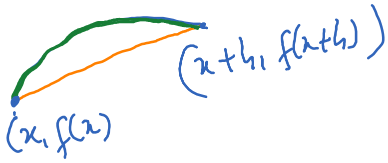


Let $r(x)$ be the length of the arc measured from $(a, f(a))$ to $(x, f(x))$. Note that $r(a)=0$. We want to know $r(b)$.

$$r(b) = r(b) - r(a) = \int_a^b r'(x)\,dx$$

We want to know what $r'(x)$ is.

$$r'(x) = \lim_{h\to 0} \frac{r(x+h) - r(x)}{h}$$



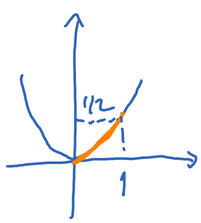
$$r(x+h) - r(x) = \text{green arc}$$
$$\approx \text{orange chord}$$
$$= \sqrt{h^2 + (f(x+h) - f(x))^2}$$
$$= h\sqrt{1 + \left(\frac{f(x+h)-f(x)}{h}\right)^2}$$

Thus, $$\frac{r(x+h) - r(x)}{h} \approx \sqrt{1 + \left(\frac{f(x+h)-f(x)}{h}\right)^2}$$

Let $h \to 0$:
$$r'(x) = \sqrt{1 + f'(x)^2}$$

Therefore, the arc on the graph of f from $(a, f(a))$ to $(b, f(b))$ is
$$\int_a^b \sqrt{1+f'(x)^2}\,dx$$

$\underline{\text{Ex}}$ Find the length of the parabola $y = \frac{x^2}{2}$ from $(0,0)$ to $(1, \frac{1}{2})$.



$$f(x) = \frac{x^2}{2}$$
$$f'(x) = x$$
$$\sqrt{1+f'(x)^2} = \sqrt{1+x^2}$$

$$\text{Arclength} = \int_0^1 \sqrt{1+x^2}\,dx$$

Let $x = \tan u$
$dx = x'\,du = \sec^2 u\,du$

x	0	1
u	0	$\frac{\pi}{4}$

Thus, $\int_0^1 \sqrt{1+x^2}\,dx = \int_0^{\pi/4} \sqrt{1+\tan^2 u}\ \sec^2 u\,du$

$$= \int_0^{\pi/4} \sec u \sec^2 u\,du = \int_0^{\pi/4} \sec^3 u\,du$$

$$= \int_0^{\pi/4} \frac{1}{\cos^3 u}\,du = \int_0^{\pi/4} \frac{1}{\cos^4 u} \cos u\,du \qquad (*)$$

Let $v = \sin u$. Then $dv = v'\,du = \cos u\,du$

$$\cos^4 u = (\cos^2 u)^2 = (1-\sin^2 u)^2 = (1-v^2)^2$$

$(*) = \int_0^{\sqrt{2}/2} \frac{1}{(1-v^2)^2}\,dv = \int_0^{\sqrt{2}/2} \frac{1}{(1-v)^2(1+v)^2}\,dv = \dots$ partial fraction decomposition