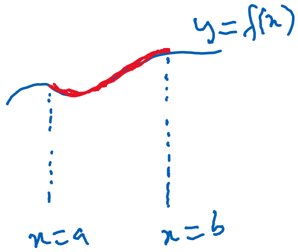
Lecture 22

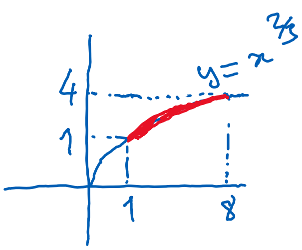
Friday, September 29, 2023 1:37 PM

Prayer



$$L = \int_a^b \sqrt{1+f'(x)^2}\,dx$$

$\underline{\underline{Ex}}$ Find the length of the graph of $y = x^{2/3}$ when $1 \le x \le 8$.



$$f(x) = x^{2/3}$$

$$f'(x) = \frac{2}{3} x^{-1/3}$$

$$f'(x)^2 = \frac{4}{9} x^{-2/3}$$

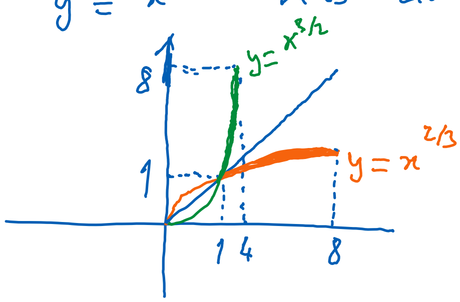
$$1 + f'(x)^2 = 1 + \frac{4}{9} x^{-2/3}$$

$$L = \int_1^8 \sqrt{1 + \frac{4}{9} x^{-2/3}}\,dx$$

This is quite a difficult integral to compute by hand.

<u>Observation</u>:

$y = x^{2/3}$ has an inverse function $y = x^{3/2}$.

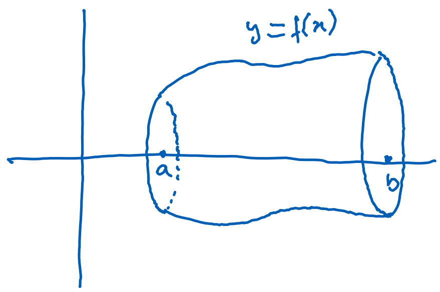


The length in orange = the length in green

$$L = \int_1^4 \sqrt{1 + \frac{9}{4} x}\,dx$$

We know how to find this integral (review last lecture).

Surface of revolution

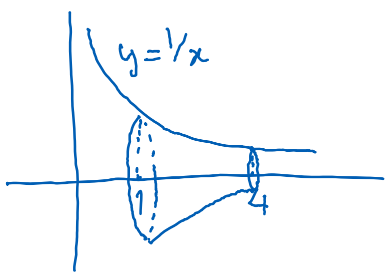


volume $= \int_a^b A(x)\,dx = \int_a^b \pi f(x)^2\,dx$

Surface (not including the top and bottom caps)

$$= \int_a^b 2\pi f(x)\sqrt{1+f'(x)^2}\,dx$$

Ex



$f(x) = \frac{1}{x}$

$f'(x) = -\frac{1}{x^2}$

$f'(x)^2 = \frac{1}{x^4}$

$$S = \int_1^4 2\pi \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx$$

$$= \int_1^4 \frac{2\pi}{x^3}\sqrt{1+x^4}\,dx$$

Let $u = x^2$

Then $du = 2x\,dx \rightsquigarrow dx = \frac{du}{2x}$

$$S = \int_1^{16} \frac{2\pi}{x^3}\sqrt{1+u^2}\,\frac{du}{2x} = \pi\int_1^{16}\frac{\sqrt{1+u^2}}{x^4}\,du = \pi\int_1^{16}\frac{\sqrt{1+u^2}}{u^2}\,du$$

Substitution: $u = \tan v$

$du = \sec^2 v\,dv$

$1+u^2 = 1+\tan^2 v = \sec^2 v$

$$\rightsquigarrow \int_1^{16}\frac{\sqrt{1+u^2}}{u^2}\,du = \int_{\pi/4}^{?}\frac{\sec v}{\tan^2 v}\sec^2 v\,dv$$

$$= \int_{\pi/4}^{?}\frac{1}{\cos v\,\sin^2 v}\,dv$$

Then use the substitution $y = \sin v$

$$dy = y' dv = \cos v \, dv$$

$$\int_{\pi/4}^{?} \frac{1}{\cos v \sin^2 v} dv = \int_{\sqrt{2}/2}^{?} \frac{1}{(\cos v) y^2} \frac{dy}{\cos v} = \int_{\sqrt{2}/2}^{?} \frac{dy}{y^2 \cos^2 v}$$

$$= \int_{\sqrt{2}/2}^{?} \frac{dy}{y^2 (1-y^2)}$$

Use partial fraction decomposition...