Lecture 36

Monday, October 23, 2023 12:49 AM

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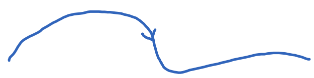
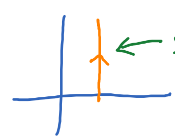
Now that we know how parametric equations work, we want to know how to do Calculus on a curve. Calculus helps us answers questions such as finding the tangent line, the speed, the length, the area enclosed by the curve.



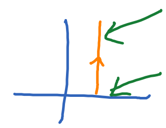
A parametrization encodes all details about the curve, not just its shape. For example, we know how fast the curve is drawn. Think about the curve as your route and the parametrization is your driving.

(1) $\begin{cases} x = 1 \\ y = t \end{cases}$ $0 \le t \le 9$



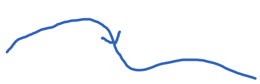
← same speed, finish in 9 seconds

(2) $\begin{cases} x = 1 \\ y = t^2 \end{cases}$ $0 \le t \le 3$



← fast here
← slow here $\Big\}$ finish in 3 seconds

Length: $s(t)$



$$s(t+\Delta t) - s(t) \approx \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$\rightsquigarrow \frac{s(t+\Delta t) - s(t)}{\Delta t} \approx \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2}$$

$$\rightsquigarrow s'(t) = \sqrt{(x')^2 + (y')^2}$$

Thus, the length of the curve $(x(t), y(t))$, $a \le t \le b$, is

$$L = s(b) - s(a) = \int_a^b s'(t)\,dt = \int_a^b \sqrt{(x')^2 + (y')^2}\,dt$$

Speed at time t is $s'(t) = \sqrt{(x')^2 + (y')^2}$.

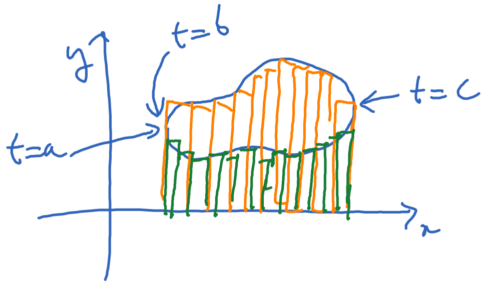
How about the tangent line?

$$r(t) = (x(t), y(t))$$

$$r'(t) = (x'(t), y'(t))$$

Area enclosed by a curve:

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases} \quad a \le t \le b$$



$$\text{Green area} = \int_{x(a)}^{x(c)} y\,dx = \int_a^c y x'\,dt$$

$$\text{Orange area} = \int_{x(c)}^{x(b)} y\,dx = \int_c^b y x'\,dt$$

Note that the green area is positive but the orange area is negative.

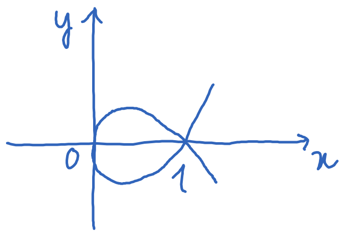
Thus, the area enclosed by the curve is

$$|\text{green} + \text{orange}| = \left| \int_a^c y x'\,dt + \int_c^b y x'\,dt \right| = \left| \int_a^b y x'\,dt \right|$$

If we swap x and y in the above argument, we also get

$$\text{enclosed area} = \left| \int_a^b y x'\,dt \right| = \left| \int_a^b x y'\,dt \right|.$$

Ex $\begin{cases} x = t^2 \\ y = t^3 - t \end{cases}$



- Find the values of t at the intersection point.
- Find the speed at the origin.
- Find the two tangent lines at the self-intersection point.
- Find the length of the closed curve (write integral only, use Mathematica to evaluate it numerically.
- Find the area enclosed by the loop of the curve.