Lab 2

In this lab, you will learn how to:

- differentiate a function,
- graph an equation,
- find the tangent line to a curve,
- find derivative of an implicitly defined function.

The next lab will be built upon this lab, so please make sure that you go through all the instruction carefully and do all the assignments.

1 To turn in

Do Problems 1-24 in a single Mathematica Notebook file, or ipynb file if you use JupyterLab. Write *your name* and *lab number* at the beginning of your report. Clearly label each problem to separate them from other problems. Make sure to comment on each problem. If your code doesn't run correctly, explain what you are trying to do. **Failed code without any comment/explanation will receive 0 point.** Submit on Canvas both the *pdf file* and the *source file* (nb or ipynb).

Problems	Points
2, 3, 8, 15, 18-21	1
1, 4-7, 9, 10, 12, 14, 22, 23	2
11, 13	3
16	4
17, 24	5
Readability of your report	3
Total: 24	Total: 53

2 Find derivative of a function

1. To find the first, second, and third order derivatives of the function $\cos(x)$, try the following:

   ```
   D[Cos[x],x]
   D[Cos[x],{x,2}]
   D[Cos[x],{x,3}]
   ```

   Define the function `f[x_]:=Cos[x]` and repeat the above commands but now replace `Cos[x]` by `f[x]`.

2. Alternatively, you can simply run `f'[x]`, `f''[x]`, `f'''[x]`.

3. To evaluate $f''(-\pi/4)$, run the command `f''[-Pi/4]`. Alternatively, you can run

   ```
   D[f[x], {x, 2}] /. {x -> -Pi/4}
   ```

   Here, the symbol "/." is called the *replacement* (or *substitution*) operator. The above command will first differentiate the function $f(x)$ twice, and then substitute $x = -\pi/4$.

4. Find the third derivative of the function $f(x) = e^{\cos(x^2)}$. Then evaluate numerically $f''(1)$ using 6 significant digits.

5. Define the piecewise function

$$f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$$

In the **Piecewise** command, the unequal sign $\neq$ is typed as `!=` and the equal sign $=$ is typed as `==`. Find $f'(x)$.

You will see that $f'(0) = 0$ according to Mathematica. Mathematica's answer that $f'(0) = 0$ is actually incorrect! One cannot differentiate a formula that is valid at only a single point; it is also necessary to understand how the function behaves in a neighborhood of this point. This example shows that you shouldn't blindly believe Mathematica although it is quite reliable most of the time.

3 Graph an equation

Suppose two quantities, x and y, are related to each other by an equation. Examples of such an equation include:

$$y = \sin(x) \tag{1}$$

$$x^2 + 2y = 5 \tag{2}$$

$$ye^{x+y} = 1 \tag{3}$$

Each pair (x, y) satisfying the equation is represented by a point on the plane with coordinates (x, y). *To graph the equation is to place all such points on the plane.* If there is a single y on the left hand side of the equation and no occurrences of y on the right hand side, such as in Equation (1), then we can use the `Plot` command to show a graph of y. In this case, graphing the equation is simply graphing a function.

6. For example, to graph Equation (1), we run:

```
Plot[Sin[x],{x,0,4*Pi}]
```

Can you graph the equation $y = \ln(e^x + 1)$, where x ranges from -5 to 5 ?

7. In Equation (2), although both x and y are on the same side of the equation, you can easily solve for y as a function of x and then graph it with `Plot`. Plot y as a function of x, where x ranges from -3 to 3.

8. In Equation (3), it is very difficult to solve for y as a function of x. In this case, we will graph the equation as it is by the command **ContourPlot**. Try the following:

```
ContourPlot[y*E^(x+y)==1,{x,-2,2},{y,0,10}]
```

Notice the double equal sign. Mathematica will try to solve numerically for the pairs (x, y), with $-2 \leq x \leq 2$ and $0 \leq y \leq 10$, that satisfy Equation (3) and then place them on the plane.

9. Graph the equation $\ln(x^2y) = \cos(xy) + 1$. Choose your own range of x and y to get a good-looking graph.

10. The following equation represents an the ellipse

$$\frac{x^2}{4} + \frac{y^2}{16} = 1$$

Draw this ellipse using `ContourPlot` with the range $-4 \leq x, y \leq 4$.

11. Equations of the form
$$y^2 = x^3 + ax + b$$
represent *elliptic curves*. Elliptic curves are an important area in Number Theory. In fact, the British mathematician Andrew Wiles used them to prove the notorious Fermat's Last Theorem. For $a = -2$ and for each value of $b = 2, 1, 0$, plot the elliptic curve where $-3 \leq x, y \leq 3$. For $a = 0$ and for each value of $b = 1, 0, -1$, plot the elliptic curve where $-3 \leq x, y \leq 3$.

4 Find the tangent line to a curve

The graph of function $f(x) = x^2$ is a parabola. The slope of the line that is tangent to the parabola at the point (x_0, y_0), where $y_0 = f(x_0)$, is $f'(x_0)$. Thus, the equation of this tangent line is
$$y = y_0 + f'(x_0)(x - x_0)$$
For example, at $(x_0, y_0) = (3, 9)$, the tangent line has the equation $y = 9 + 6(x - 3) = 6x - 9$.

12. You can draw the function and the tangent line together on the same plot by

```
Plot[{x^2, 6 x - 9}, {x, 0, 5}]
```

Add the option `PlotLegends -> "Expressions"` to the above command.

13. Define the function $f(x) = \frac{x}{\sqrt{x^2+1}}$. On the same plot, graph the function f together with the two lines tangent to its graph at $x_0 = 1$ and $x_0 = -1$.

In the next two problems (Problem 14-15),
$$f(x) = \frac{2x + 1}{x^2 + 1}$$
and our goal is to find the values of x at which the tangent line is horizontal.

14. Define the function f and plot it. How many places on the graph is the tangent line horizontal? Give a rough estimate of those values of x.

15. Recall that the slope of the tangent line is $f'(x)$. For the tangent line to be horizontal, we need its slope to be zero, i.e. $f'(x) = 0$. To solve for the exact solutions x of this equation, we use the command **Solve** as follows.

```
Solve[f'[x] == 0, x]
```

To get numerical values of x, use the command **NSolve** instead of **Solve**.

```
NSolve[f'[x] == 0, x]
```

16. Plot the function $f(x) = x^5 + x^4 - 4x^3 - 3x^2 + 3x + 1$ on the interval $[-2, 2]$. Find all the values of x at which the tangent line to the graph of f is horizontal. Then plot these tangent lines together with the graph of f on the same plot.

17. Let
$$f(x) = \frac{8x}{\sqrt{x^4 + x + 1}}$$

(a) Plot the function f on the interval $[-5, 5]$.

(b) Add the option `AspectRatio -> Automatic` to make sure that the x-axis and y-axis have the same scaling.

(c) Then add the option `AxesOrigin -> {0, 0}` to make sure that the x-axis and y-axis intersect each other at the origin $(0, 0)$.

(d) Find all the values of x at which the tangent line to the graph of f has slope equal to -1. *Note: when solving the equation $f'(x) = 0$, you will get some complex-valued roots. Pick only the real-valued roots.*

(e) Then plot these tangent lines together with the graph of f on the same plot.

5 Implicit differentiation

Suppose two quantities x and y are related to each other by an equation

$$2(x^2 + y^2 - x)^2 = 25(x^2 - y^2) \tag{4}$$

Let us view y as a function of x. Our goal is to find the derivative $y'(x)$ in terms of x and $y(x)$. As a matter of convenience, the notations $y'(x), x, y(x)$ are sometimes shortened as y', x, y.

18. First, we give the equation a name, such as `eq`, as follows.

```
eq = 2 (x^2 + y^2 - x)^2 == 25 (x^2 - y^2)
```

The equal sign "=" is the *assignment operator*, which assigns the right hand side to the left hand side. The double equal sign "==" is to represent an *equation*, not an assignment.

19. Next, we replace y by $y(x)$ (viewing y as a function of x) and then differentiate both sides of `eq` by x. We get a new equation, which we name `deq`.

```
deq = D[eq /. {y -> y[x]}, x]
```

20. Then we solve for $y'(x)$ from the equation `deq`.

```
Solve[deq, y'[x]]
```

You should get

$$y'(x) = \frac{-4x^3 + 6x^2 - 4xy(x)^2 + 2y(x)^2 + 23x}{y(x)\,(4x^2 + 4y(x)^2 - 4x + 25)}$$

21. Starting from Mathematica 13.1, the procedure of finding implicit derivative above (Problems 18-20) can be shortened by the following command:

```
ImplicitD[2 (x^2 + y^2 - x)^2 == 25 (x^2 - y^2), y, x]
```

22. Plot the equation (4) using **ContourPlot**. What shape does the curve look like?

23. To find the location on the curve where the tangent line is horizontal, we find all the points (x, y) on the curve such that $y' = 0$. Therefore, we need to solve two equations

$$-4x^3 + 6x^2 - 4xy^2 + 2y^2 + 23x = 0, \quad 2(x^2 + y^2 - x)^2 = 25(x^2 - y^2)$$

for two unknowns x and y. To do so, we use the **Solve** (or **NSolve**) command as follows.

```
NSolve[{23 x + 6 x^2 - 4 x^3 + 2 y^2 - 4 x y^2 == 0,
            2 (x^2 + y^2 - x)^2 == 25 (x^2 - y^2)}, {x, y}]
```

You will see some pairs (x, y) that are complex valued. Only pick the pairs (x, y) where both x and y are real. These pairs are the positions at which the tangent line are horizontal. How many of such pairs do you get? Does it agree with the graph of the equation?

24. Consider the equation $-4y^3 + 3y^2 - y^5 = x^4 - 3x^3 + 2x^2$.

    (a) Plot this equation. By observing the picture, how many positions (x, y) are there on the curve where the tangent line is horizontal? How about vertical?

    (b) Viewing y as a function of x, find y'. Note that the product xy should be typed as `x*y` or `x y` (space between `x` and `y`), not `xy`.

    (c) Viewing x as a function of y, find x'.

    (d) Find all the positions (x, y) on the curve at which the tangent line is horizontal.

    (e) Find all the positions (x, y) on the curve at which the tangent line is vertical.
    Hint: $x' = 0$