Lab 3

In this lab, you will learn how to:

- find local extrema (local maxima/minima) of a function,
- find absolute extrema of a function,
- find critical points and inflection points,
- solve an equation using Newton's method.

The next lab will be built upon this lab, so please make sure that you go through all the instruction carefully and do all the assignments.

1 To turn in

Do Problems 1-27 in a single Mathematica Notebook file, or ipynb file if you use JupyterLab. Write *your name* and *lab number* at the beginning of your report. Clearly label each problem to separate them from other problems. Make sure to comment on each problem. If your code doesn't run correctly, explain what you are trying to do. **Failed code without any comment/explanation will receive 0 point.** Submit on Canvas both the *pdf file* and the *source file* (nb or ipynb).

Problems	Points
3, 5, 8-10, 15, 22, 23	1
1, 2, 6, 7, 14, 16-20, 24-26	2
11, 12, 27	3
4, 13	4
21	6
Readability of your report	3
Total: 27	Total: 60

2 Find local extrema of a function

On the graph of a function, local maxima and local minima (collectively called *local extrema*) are the locations of the 'peaks' and 'valleys'. Keep in mind that a local extremum is a point with x, y coordinates and not just a number. For example, if the point $(3, 5)$ is a valley on the graph, we say that *the function attains local minimum 5 at* $x = 3$.

1. Define and graph the function

$$f(x) = \frac{x^2}{3} + (2^x - 20)\sin(x)\cos(x^2) \tag{1}$$

   on the interval $[0, 5]$.

2. How many local maxima and how many local minima are there on the graph? (Note that the endpoints of the graph do not count as local maxima or local minima.) Based on the graph, estimate roughly the positions (x, y coordinates) of local maxima and local minima.

The commands **FindMaximum** and **FindMinimum** search the local maxima and local minima of a function. These two commands have exactly the same syntax.

- Use `FindMaximum[f[x],{x,c}]` to find a local maximum located near $x = c$. Mathematica will *attempt* to search for such a local maximum. It might return a local maximum that is quite far away from c. It is important that you double check the result with the graph.

- Use `FindMaximum[{f[x],cond},{x,c}]` to find a local maximum located near $x = c$ under a certain condition `cond`. For example, if you only want $x \in [a, b]$ then the condition `cond` will be `a<=x<=b`.

3. To find the local maximum located near $x = 3$ of the function f given by Equation (1), try the command

```
FindMaximum[{f[x], 0 <= x <= 5}, {x, 3}]
```

Try removing the condition $x \in [0, 5]$ from the above command and see if the result is still the same.

4. Use `FindMaximum` and `FindMinimum` commands to find the positions (x, y coordinates) of all local maxima and local minima of the function f on the interval $[0, 5]$. The endpoints of the interval are not counted as local extrema.

A point on the plane may be viewed as a pair of numbers: the x-coordinate and the y-coordinate. You are familiar with the notation (x, y). In Mathematica, you use the *list notation* $\{\}$ to represent a point. For example, $\{2, 5\}$ represents the point with coordinates $x = 2, y = 5$. The notation `{{1,2},{2,-1},{3,0},{2,1}}` represents a list of four points on the plane with coordinates $(1, 2), (2, -1), (3, 0), (2, 1)$. To graph the list a points, we use the command **ListPlot**.

5. To graph the list of points `{{1,2},{2,-1},{3,0},{2,1}}`, use the command:

```
ListPlot[{{1,2},{2,-1},{3,0},{2,1}}]
```

To change the color of the points, you can add a style option as follows:

```
ListPlot[{{1,2},{2,-1},{3,0},{2,1}},PlotStyle->Red]
```

You can also customize the size of the points. Try replacing the `PlotStyle` option above with `PlotStyle->{Red,PointSize[0.02]}`.

6. Using the command **Show**, you can put together a list a points and a graph of a function, say $y = 2x^2 - 9x + 9$, on the same plot. Try the following:

```
p1 = ListPlot[{{1, 2},{2, -1},{3, 0},{2, 1}},PlotStyle->{Red,PointSize[0.02]}];
p2 = Plot[2 x^2 - 9 x + 9, {x, 0.7, 3.5}];
Show[p1, p2]
```

Put the option `AspectRatio->Automatic` inside the command `Show` to make sure that the x-axis and the y-axis have the same scaling. Try interchanging `p1` and `p2` in the command `Show`. What difference do you see?

7. Let f be the function given by Equation (1). Draw the graph of f on $[0, 5]$ and put all the local extrema you found in Problem 4 on the the graph of f.

3 Find absolute extrema of a function

The commands **Maximize** and **Minimize** returns the absolute maximum and minimum *in exact form* of a function. These two commands have exactly the same syntax.

- `Maximize[f[x],x]` returns the maximum value of f and one value of x where the maximum is attained.
- `Maximize[{f[x], a<=x<=b}, x]` returns the maximum value of f on the interval $[a, b]$ and one value of x where the maximum is attained.

Make sure to always double check the result with the graph because Mathematica may return a wrong answer, especially when min/max doesn't exist. When the exact form of the max/min is hard to find, **Maximize** and **Minimize** may not return anything. In that case, replace them with **NMaximize** and **NMinimize** to find max/min *in numerical form.*

8. Try the following command:

   ```
   Minimize[Sin[x], x]
   ```

   Do you get *all* the values of x where the sine function attains its minimum value?

9. Try the following command:

   ```
   Minimize[{Sin[x], 0 <= x <= 2 Pi}, x]
   ```

   Does the result agree with the graph?

10. Try the following command

    ```
    Minimize[{Sin[x], 0 < x < Pi}, x]
    ```

    Does the result agree with the graph? Does the sine function have a minimum value for $x \in (0, \pi)$?

11. Let $f(x) = \frac{x}{x^4+1}$.
    (a) Graph the function f.
    (b) Find the absolute maximum and minimum value of f for $x \in \mathbb{R}$.
    (c) Find the absolute maximum and minimum value of f for $x \in [0.1, 2]$. Now replace 0.1 by $\frac{1}{10}$ and run the commands again. Do you see any difference?

12. Let $f(x) = \frac{x}{\cos(x^2)+2}$.
    (a) Graph the function f for $x \in [\sqrt{2}, \sqrt{8}]$.
    (b) Find the absolute maximum and minimum value of f on the interval $[\sqrt{2}, \sqrt{8}]$.

13. As you can see in Problem 6, the point $(2, 1)$ doesn't lie on the parabola $y = 2x^2 - 9x + 9$. The distance from $(2, 1)$ to a point (x, y) on the parabola is

    $$d = \sqrt{(x-2)^2 + (y-1)^2}$$

    Find the point on the parabola that has the shortest distance to $(2, 1)$. Find this shortest distance. Draw this point together with $(2, 1)$ and the parabola on the same plot. *Hint:* view the distance d as a function of x and try to minimize d. You might need to try `N[Minimize[...]]` instead of `NMinimize`.

4 Find critical points and inflection points

In Section 2, you learned how to find local extrema using the `FindMaximum` and `FindMinimum` commands. The local extrema are critical numbers of a function. But not all critical numbers are local extrema. Recall that c is a critical number of f if $f'(c)$ does not exist or if it exists and is equal to zero. Therefore, the problem of finding critical numbers is essentially the problem of solving the equation $f'(x) = 0$. Likewise, the problem of finding inflection points is essentially the problem of solving the equation $f''(x) = 0$.

To solve an equation, you can use the command **Solve** (to solve exactly) or **NSolve** (to solve numerically).

14. Define and graph the function $f(x) = x^6 - 5x^5 - x^4 + 21x^3$ on the interval $[-2, 4]$. How many critical numbers are there on the graph? Does f attain a local extremum at every critical number?

15. Find all critical numbers using the command

    ```
    Solve[f'[x] == 0, x]
    ```

    Then try replacing `Solve` with `NSolve`.

16. A *critical point* is a point on the graph whose x-coordinate is a critical number. Find all critical points of the function f.

17. On what subintervals of $(-\infty, \infty)$ is the function increasing? decreasing?

18. Find all inflection points using the command

    ```
    NSolve[f''[x] == 0, x]
    ```

    Recall that an inflection point on the graph is where f transitions from being concave upward to concave downward or vice versa. Among the solutions you found, which of them correspond to an inflection point? Make sure that you also find the corresponding y-coordinates because an inflection point is a point, not just a number.

19. On what subintervals of $(-\infty, \infty)$ is the function concave upward? concave downward?

20. Put all the critical points and inflection points on the graph of f, with critical points as red dots and inflection points as black dots. *Hint:* use `ListPlot` to draw all critical points with red color, another `ListPlot` to draw all inflection points with black color, and then use `Show` to put them all together on the graph.

21. Let $f(x) = \frac{|x|}{x^2-x+1}$. An equivalent way to write f is

    $$f(x) = \begin{cases} \frac{x}{x^2-x+1} & \text{if } x \geq 0 \\ \frac{-x}{x^2-x+1} & \text{if } x < 0 \end{cases}$$

    (a) Use the **Piecewise** command (see Lab 1, Problem 25) to define the function f. Then graph it on the interval $[-3, 3]$.
    (b) Find all critical points (not just critical numbers) of f.
    (c) On what subintervals of $(-\infty, \infty)$ is the function increasing? decreasing?
    (d) Find all inflection points of f.
    (e) On what subintervals of $(-\infty, \infty)$ is the function concave upward? concave downward?
    (f) Put all the critical points and inflection points on the graph of f, with critical points as red dots and inflection points as black dots.

5 Solve an equation using Newton's method

By this point, you are familiar with using `Solve` and `NSolve` to solve an equation. They have worked nicely so far. However, the algorithms that are built in these commands are *not* sufficient to solve every equation. Let us take a look at an example.

22. Graph two functions e^x and $3x^2 - 1$ on the same plot. Make sure that the picture show 3 points where the two graphs meet.

23. To find the x-coordinates of these intersection points, try solving the equation $e^x = 3x^2 - 1$ using the `Solve` and `NSolve` commands. Can they solve the equation?

Newton's method is a very elegant method to solve almost any equation. The idea is simple: starting with an initial guess x_1 for solution of $f(x) = 0$, you expect better and better approximation of the solution as you compute x_2, x_3,... from the recursive formula $x_{n+1} = R(x_n)$, where

$$R(x) = x - \frac{f(x)}{f'(x)}.$$

Observe that

$$\begin{aligned} x_2 &= R(x_1) \\ x_3 &= R(x_2) = R(R(x_1)) \\ x_4 &= R(x_3) = R(R(R(x_1))) \\ x_5 &= R(x_4) = R(R(R(R(x_1)))) \end{aligned}$$

Therefore, x_k can be obtained directly from x_1 by successively applying R to it $(k - 1)$ times.

24. Let us consider an example: $f(x) = x^3 - 3x^2 - 9x + 8$. We want to find a root of the polynomial f. Define the functions f and R as follows:

    ```
    f[x_] := x^3 - 3 x^2 - 9 x + 8
    R[x_] := x - f[x]/f'[x]
    ```

    You need to pick an initial guess x_1 and the number of iteration n. For example,

    ```
    x1 = -20;
    n = 12;
    ```

    Because x_k is obtained from x_1 by applying R to it $(k - 1)$ times, you define x_k as follows:

    ```
    x[k_] := NumberForm[N[Nest[R, x1, k - 1]], 12]
    ```

    The number 12 is the number of significant digits (not the number of decimal places), which can be adjusted depending on the precision requirement. Finally, you create a table of two columns: one column for the index k and one column for x_k.

    ```
    Table[{k, x[k]}, {k, 1, n}] // TableForm
    ```

    What is an approximate value of the root correct to 10 decimal places?

25. Graph the polynomial f. How many roots does it have? Does the root you obtained in the previous problem agree with the graph?

26. Alter the initial guess x_1 to find other roots of the polynomial f correct to 10 decimal places.

27. Use Newton's method with 8-decimal-place precision to find all intersection points of the graph of e^x and the graph of $3x^2 - 1$. Make sure you find both x, y coordinates.