Lecture 17

Tuesday, October 1, 2024 3:09 AM

The motion problem:
Imagine an object moving in one direction with position function s(t). The instantaneous velocity is

$$v(t) = \lim_{h\to 0} \frac{s(t+h)-s(t)}{h} = s'(t)$$

The acceleration is

$$a(t) = \lim_{h\to 0} \frac{v(t+h)-v(t)}{h} = v'(t) = s''(t)$$

Recall the derivatives of some functions we have already found:

$f(x) = x^2$ has derivative $f'(x) = 2x$

$f(x) = x$ has derivative $f'(x) = 1$

$f(x) = c$ has derivative $f'(x) = 0$

$f(x) = 1/x$ has derivative $f'(x) = -1/x^2$

Relation between differentiability and continuity: If a function is differentiable at a then it is continuous at a. But the converse is not true (Think about the absolute value function.)

Algebraic rules to find derivatives:

The sum law: $(f+g)' = f' + g'$

Why? Let $h(x) = f(x) + g(x)$. Then

$$h'(x) = \lim_{t\to x} \frac{h(t)-h(x)}{t-x} = \lim_{t\to x} \frac{f(t)+g(t)-\big(f(x)+g(x)\big)}{t-x} = \lim_{t\to x}\left(\frac{f(t)-f(x)}{t-x} + \frac{g(t)-g(x)}{t-x}\right)$$
$$= f'(x) + g'(x)$$

The scaling law: $(c\,f)' = c\,f'$ where c is a constant.

Ex:

Find the derivative of $f(x) = x^2 - 2x + 3$

Here, the function f is the sum of three functions: $x^2, -2x, 3$.

$f'(x) = (x^2)' + (-2x)' + (3)' = 2x + (-2)x' + 0 = 2x - 2$

Ex:

Find the derivative of $f(x) = \frac{x^2-3x+2}{x}$

The product law: $(fg)' = f'g + g'f$

Notice that $(fg)' \neq f'g'$ (this is a very common mistake)

Why? Let $h(x) = f(x)g(x)$. Then

$$h'(x) = \lim_{t\to x} \frac{h(t)-h(x)}{t-x} = \lim_{t\to x} \frac{f(t)g(t)-f(x)g(x)}{t-x}$$

$$\frac{h(t)-h(x)}{t-x} = \frac{f(t)g(t)-f(x)g(x)}{t-x} = \frac{f(t)\big(g(t)-g(x)\big)+g(x)(f(t)-f(x))}{t-x}$$
$$= f(t)\frac{g(t)-g(x)}{t-x} + g(x)\frac{f(t)-f(x)}{t-x}$$

With the product rule, we can find the derivative of any polynomials:

Ex:
$x^3 = (x^2)x$
So, $(x^3)' = (x^2)'x + x^2(x') = (2x)x + x^2 = 3x^2$

Ex:
$x^4 = (x^3)x$
So, $(x^4)' = (x^3)'x + x^3(x') = (3x^2)x + x^3 = 4x^3$

More generally, $(x^n)' = nx^{n-1}$ for any positive integer n.