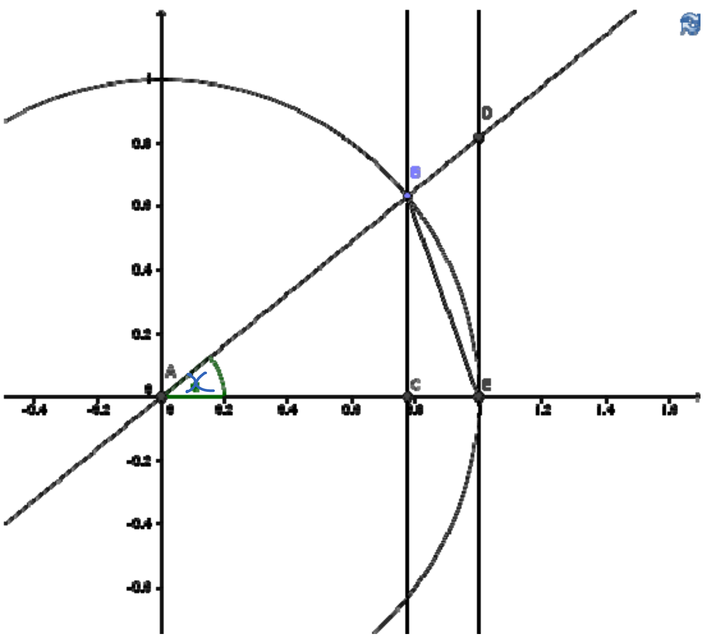
Lecture 20

Monday, October 7, 2024 5:08 AM

Today, we will find derivative of a function involving trigonometric functions.
How do we differentiate the function $\sin(x)$? The only tool at hand for us to use is the definition of derivative as a limit of difference quotient.

$$(\sin x)' = \lim_{h\to 0} \frac{\sin(x+h)-\sin x}{h} = \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$
$$= \sin x \lim_{h\to 0} \frac{\cos h - 1}{h} + \cos x \lim_{h\to 0} \frac{\sin h}{h}$$

An important limit is know is $\lim_{h\to 0} \frac{\sin h}{h} = 1$. There is a neat geometric explanation for this fact.



length of arc BE is x.

$AC = \cos x$

$AB = AE = 1$

$BC = \sin x$

$DE = \tan x$

area of sector ABE is $\frac{x(\pi)}{2\pi} = \frac{x}{2}$

area of triangle ABE is $\frac{1}{2} AE \cdot BC = \frac{1}{2}\sin x$

(Image from https://math.stackexchange.com/questions/907362/why-the-limit-of-frac-sinxx-as-x-approaches-0-is-1)

area of triangle ADE is $\frac{1}{2} AE \cdot DE = \frac{1}{2}\tan x$.

Thus,

$$\frac{1}{2}\sin x \le \frac{1}{2} x \le \frac{1}{2}\tan x$$

Divide the inequality by $x > 0$:

$$\begin{cases} \frac{\sin x}{x} \le 1 \\ \frac{\sin x}{x} \geq \cos x \end{cases}$$

By Squeeze's theorem,

$$\lim_{x\to 0^+} \frac{\sin x}{x} = 1$$

Reason similarly to get $\lim_{x \to 0^-} \frac{\sin x}{x} = 1.$

The derivative of cos(x) can be achieved using the definition of derivative.

$(\cos x)' = \lim_{h\to 0} \frac{\cos(x+h)-\cos x}{h} = \lim_{h\to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h}$
$= \cos x \lim_{h\to 0} \frac{\cos h - 1}{h} - \sin x \lim_{h\to 0} \frac{\sin h}{h} = (\cos x)0 - (\sin x)1 = -\sin x$

Ex: find the derivatives of $\tan x$, $\cot x$, $\sec x$, $\csc x$.

Ex: find the limit
$$\lim_{x\to 0} \frac{\sin(3x)\tan(5x)}{2x^2}$$