Lecture 4

Monday, September 9, 2024 4:23 AM

* Raising to power: $2^n = 2 \cdot 2 \cdots 2$ (there are n number 2's)

$$2^1 = 2$$
$$2^3 = 2 \cdot 2 \cdot 2 = 8$$
$$2^0 = 1 \quad \text{(convention)}$$

How about 2^x where x is not a natural number?

$$2^{-3} = \frac{1}{2^3}, \quad 2^{-n} = \frac{1}{2^n}$$

$$2^{p/q} = \sqrt[q]{2^p}, \quad 2^{\frac{1}{3}} = \sqrt[3]{2^1} = \sqrt[3]{2}, \quad 2^{-2/3} = \frac{1}{2^{2/3}} = \frac{1}{\sqrt[3]{2^2}} = \frac{1}{\sqrt[3]{4}}$$

$2^{\sqrt{2}} = ?$ At this point, this is a question about possibility, not about the most efficient way to compute $2^{\sqrt{2}}$.

$\sqrt{2} \approx 1.4142135624$

$$2^{\sqrt{2}} \approx 2^1 = 2$$
$$2^{\sqrt{2}} \approx 2^{1.4} = 2^{\frac{14}{10}} = 2^{\frac{7}{5}} = \sqrt[5]{2^7} = \sqrt[5]{128}$$
$$2^{\sqrt{2}} \approx 2^{1.41} = 2^{\frac{141}{100}} = \sqrt[100]{2^{141}}$$
$$2^{\sqrt{2}} \approx 2^{1.414} = 2^{\frac{1414}{1000}} = 2^{\frac{707}{500}} = \sqrt[500]{2^{707}}$$

.. ....

Each time we go, we get a better and better approximation of $2^{\sqrt{2}}$.

In general, one can compute, at least in theory, b^x for any $b > 0$ and $x \in \mathbb{R}$.

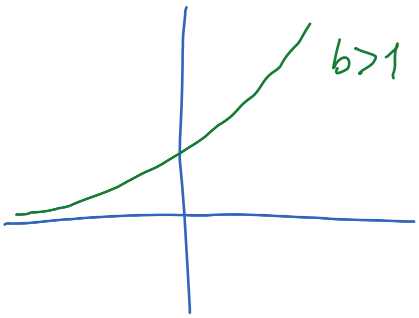
exponents

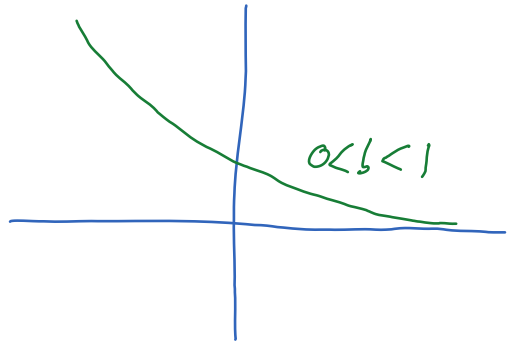
base

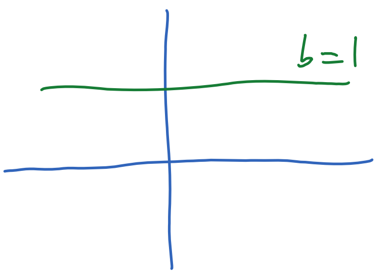
You will run into trouble if you try to compute $(-2)^{\sqrt{2}}$ because

$$(-2)^{\sqrt{2}} \approx (-2)^{1.41} = \sqrt[100]{\underbrace{(-2)^{141}}_{\text{negative}}} \longrightarrow \text{can't take } 100^{th} \text{ root of a negative number.}$$

The graph of the function $f(x) = b^x$ looks like:





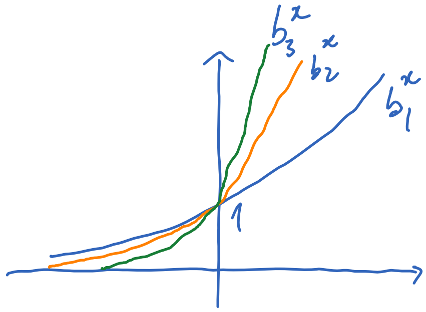


We are only interested in the case $b \neq 1$.

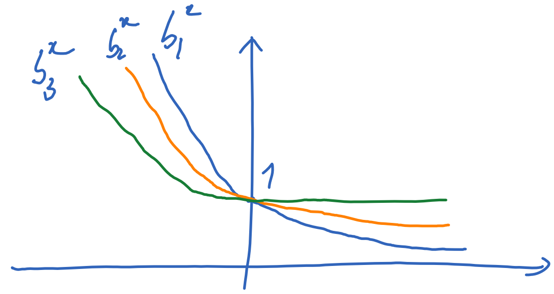
Note the behaviors:

$b>1$:
- b^x is an rapidly increasing function
- b^x decays to 0 as x goes to $-\infty$
- b^x grows to ∞ as x goes to ∞
- the y-intercept is $(0,1)$

$0<b<1$:
- .... decreasing..
- - - - ... ... ∞
- - - ---- $-\infty$
- - - - - - - - -



$1 < b_1 < b_2 < b_3$



$0 < b_1 < b_2 < b_3 < 1$

For the slope of the tangent line to the curve at $(0,1)$ to be 1,

$b \approx \underbrace{2.71828....}$

This number will be designated "e" (the Neper number)

$e^x \approx 2.71828^x$

* <u>Law of exponents:</u>

Sum : $b^{x+y} = b^x b^y$

Difference: $b^{x-y} = \dfrac{b^x}{b^y}$

product: $b^{xy} = (b^x)^y$

product (in base): $(ab)^x = a^x b^x$

Practice on law of exponents on the worksheet.