Lecture 10

Friday, September 27, 2024 9:24 AM

* Cauchy sequence: how do we prove that a Cauchy sequence must converge?

This is probably the most useful property of a Cauchy sequence. There are interesting techniques in the proof that are worth learning.

* Our strategy is as follows: first, we show that a Cauchy sequence must be bounded. As a consequence of Bolzano-Weierstrass theorem, this sequence must have a convergent subsequence (a_{n_k}) : $\lim a_{n_k} = a$. Then we will show that $\lim a_n = a$.

Ex Sequence $a_n = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}$

Show that (a_n) converges.

There are two ways to prove it.

Method 1: observe that (a_n) is an increasing sequence.

If you can prove that it is bounded from above, then you will know that it must have a limit. Notice that

$$\frac{1}{k^2} < \frac{1}{k^2-k} = \frac{1}{k(k-1)} = \frac{1}{k-1} - \frac{1}{k}$$

Thus,

$$\left.\begin{aligned} \frac{1}{2^2} &< \frac{1}{1} - \frac{1}{2} \\ \frac{1}{3^2} &< \frac{1}{2} - \frac{1}{3} \\ &\cdots \\ \frac{1}{n^2} &< \frac{1}{n-1} - \frac{1}{n} \end{aligned}\right\} \quad \frac{1}{2^2} + \cdots + \frac{1}{n^2} < \frac{1}{1} - \frac{1}{n} = 1 - \frac{1}{n} \quad \text{for any } n \geq 2.$$

Thus, $a_n = 1+\frac{1}{2^2}+\dots+\frac{1}{n^2} < 1+\left(1-\frac{1}{n}\right) = 2-\frac{1}{n} < 2$ for all $n \geq 2$.

Therefore, (a_n) is bounded from above.

Method 2: show that (a_n) is a Cauchy sequence

For $m > n$:

$$a_m - a_n = \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \dots + \frac{1}{m^2} < \left(\frac{1}{n}-\frac{1}{n+1}\right)+\left(\frac{1}{n+1}-\frac{1}{n+2}\right)+\dots+\left(\frac{1}{m-1}-\frac{1}{m}\right)$$

$$= \frac{1}{n}-\frac{1}{m}$$

Fix $\varepsilon > 0$, we need to find $N(\varepsilon)$ such that $|a_m - a_n| < \varepsilon$ for all $m, n \geq N(\varepsilon)$. We can suppose $m > n$:

$$a_m - a_n = \frac{1}{n}-\frac{1}{m} = \frac{m-n}{mn} > 0$$

$$a_m - a_n < \frac{1}{n} < \varepsilon \quad \text{provided that } n > \frac{1}{\varepsilon}$$

Choose $N(\varepsilon) = \left[\frac{1}{\varepsilon}\right]+1$.

Ex Show that the sequence $a_{n+1} = \cos(a_n)$ converges (for any initial term a_1).

$$a_{n+1} - a_n = \cos(a_n) - \cos(a_{n-1}) = 2\sin\frac{a_n - a_{n-1}}{2}\cos\frac{a_n+a_{n-1}}{2}$$

Note that $a_n \in [-1,1]$ for all $n \geq 2$. Then $a_n = \cos(a_{n-1}) \in [\cos 1, 1]$ for $n \geq 3$.

Then $a_n = \cos(a_{n-1}) \in [\cos 1, \cos(\cos 1)]$ for any $n \geq 4$.

$$\frac{a_n + a_{n-1}}{2} \in [\cos 1, \cos(\cos 1)] \text{ for any } n \geq 5.$$

Then $\quad \left|\cos\left(\frac{a_n+a_{n-1}}{2}\right)\right| = \cos\left(\frac{a_n+a_{n-1}}{2}\right) \leq \underbrace{\cos 1}_{c} < 1 \quad \forall n \geq 5$

We will use the inequality $|\sin x| \leq |x|$ for any $x \in \mathbb{R}$.

$$|a_{n+1}-a_n| = 2\left|\sin\frac{a_n-a_{n-1}}{2}\right|\left|\cos\left(\frac{a_n+a_{n-1}}{2}\right)\right| \leq 2\left|\frac{a_n-a_{n-1}}{2}\right| c = c|a_n-a_{n-1}|$$

A sequence that satisfies $|a_{n+1}-a_n| \leq c|a_n-a_{n-1}|$ is called a contractive sequence. It is a Cauchy sequence (good exercise) and thus has a limit.