Lecture 11

Monday, September 30, 2024 9:19 AM

<u>Series</u> $\sum_{n=1}^{\infty} a_n = a_1+a_2+a_3+\dots = \lim_{n\to\infty} (\underbrace{a_1+\dots+a_n}_{S_n})$

S_n is called the n'th partial sum.

The series is said to converge if the sequence (S_n) converges (or equivalently, if (S_n) is a Cauchy sequence).

<u>Ex</u> the series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots = \sum_{n=1}^{\infty}\frac{1}{n}$

$$S_n = 1+\frac{1}{2}+\dots+\frac{1}{n}$$

$$S_m = 1+\frac{1}{2}+\dots+\frac{1}{n}+\frac{1}{n+1}+\dots+\frac{1}{m}$$

$$S_m - S_n = \frac{1}{n+1}+\dots+\frac{1}{m} \quad \text{for } m>n$$

$$> \underbrace{\frac{1}{m}+\dots+\frac{1}{m}}_{m-n \text{ times}} = \frac{m-n}{m}$$

So, $S_m - S_n > 1-\frac{n}{m}$ for all $m>n$

Consequently,

$$S_{2n}-S_n > 1-\frac{n}{2n} = \frac{1}{2} \quad \forall n\in\mathbb{N}$$

Thus, (S_n) is not a Cauchy sequence. The given series (called harmonic series) diverges.

Ex $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \lim_{n\to\infty} \underbrace{\left(\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2}\right)}_{S_n}$

(S_n) is an increasing sequence and

$$S_m - S_n = \frac{1}{(n+1)^2} + \cdots + \frac{1}{m^2} < \frac{1}{n(n+1)} + \cdots + \frac{1}{(m-1)m}$$

$$= \left(\frac{1}{n} - \frac{1}{n+1}\right) + \cdots + \left(\frac{1}{m-1} - \frac{1}{m}\right)$$

$$= \frac{1}{n} - \frac{1}{m}$$

(S_n) is a bounded sequence (and also a Cauchy sequence). Thus, it converges.

* Absolute convergence and conditional convergence.

If $\displaystyle\sum_{n=1}^{\infty} |a_n| < \infty$ then $\displaystyle\sum_{n=1}^{\infty} a_n$ absolutely converges.

If $\sum |a_n| = \infty$ and $\sum a_n$ converges, we say that $\sum a_n$ conditionally converges.

* Comparison principle:

If $|a_n| \le b_n$ and $\sum b_n$ converges then $\sum a_n$ absolutely converges.

If $a_n \ge b_n > 0$ and $\sum b_n = \infty$ then $\sum a_n$ diverges.

* Ratio test with limsup:

$$L = \limsup_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$$

If $L<1$: $\sum a_n$ absolutely converges

If $L>1$: $\sum a_n$ diverges.

* Root test with limsup:

$$r = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$$

If $r<1$: $\sum a_n$ absolutely converges

If $r>1$: $\sum a_n$ diverges

To give a proof of either Ratio test or Root test, note the following property:

<u>Lemma</u> If $\limsup a_n = a$ and $b>a$ then there exists an index N such that $a_n < b$ for all $n \geq N$.

The spirit of Ratio test and Root test is to compare the series to a geometric series. Soft argument for Ratio test:

If $\limsup \left|\frac{a_{n+1}}{a_n}\right| = 0.9 < 1$

then $\exists N : \left|\frac{a_{n+1}}{a_n}\right| < 0.95 \quad \forall n > N$

otherwise, there will be a subsequence of $\left(\frac{a_{n+1}}{a_n}\right)$ that is ≥ 0.95.

That would cause $\limsup \geq 0.95$ (contradiction).

$$|a_n| = |a_1| \underbrace{\frac{|a_2|}{|a_1|}}_{\sim 0.95} \underbrace{\frac{|a_3|}{|a_2|}}_{\sim 0.95} \underbrace{\frac{|a_4|}{|a_3|}}_{\sim 0.95} \cdots \underbrace{\frac{|a_n|}{|a_{n-1}|}}_{\sim 0.95} \sim (0.95)^n$$