Lecture 12

Wednesday, October 2, 2024 9:50 AM

* Application of Ratio/Root test:

$$\sum_{n=1}^{\infty} (\sin n)^n \text{ converges or diverges?}$$

Here, $a_n = (\sin n)^n$

* Try Root Test: $|a_n| = |\sin n|^n$

$$\sqrt[n]{|a_n|} = |\sin n|$$

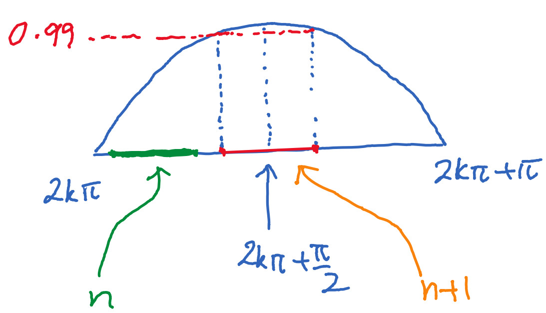
We know that $\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \limsup_{n\to\infty} |\sin n| = 1$. So, Root Test is not applicable.

* Try Ratio Test:

$$\frac{a_{n+1}}{a_n} = \frac{(\sin(n+1))^{n+1}}{(\sin n)^n} = \left(\frac{\sin(n+1)}{\sin n}\right)^n \sin(n+1)$$

Because $\limsup \sin n = 1$, there exist infinitely many n's such that

$$\sin(n+1) > 0.99$$



For those n's,

$$2k\pi + \arcsin(0.99) < n+1 < 2\pi k + \pi - \arcsin(0.99)$$

In decimal,

$$2k\pi + 1.429 < n+1 < 2k\pi + 1.712$$

or equivalently,

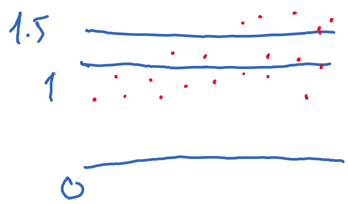
$$2k\pi + 0.429 < n < 2k\pi + 0.712$$

Thus, $\dfrac{\sin(n+1)}{\sin n} > \dfrac{0.99}{\sin(0.712)} \approx 1.515$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\sin(n+1)}{\sin n}\right|^n \sin(n+1) > 1.515^n \times 0.99 \geq 1.515 \times 0.99 > 1.5$$

We have shown that there are ∞-many n's such that

$$\left|\frac{a_{n+1}}{a_n}\right| > 1.5$$



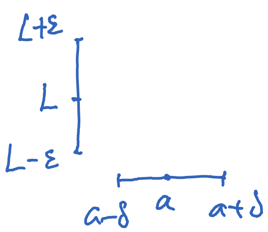
Thus, $\limsup \left|\frac{a_{n+1}}{a_n}\right| \geq 1.5 > 1$

By Ratio Test, the series $\sum a_n$ diverges.

Limit of a function

$\lim_{x\to a} f(x) = L$ if the following is true:

$\forall \varepsilon > 0, \exists \delta > 0 : |f(x) - L| < \varepsilon$ if $0 < |x-a| < \delta$.



This definition has a nickname "$\varepsilon\delta$ definition".

* Equivalent definition:

$\lim_{x\to a} f(x) = L$ if and only if:

$\forall (x_n) : x_n \to a$ and $x_n \neq a$ for all n, $f(x_n) \to L$

This is called the sequence definition of limit of functions.