# Lecture 14

Monday, October 7, 2024 9:14 AM

Use the $\varepsilon$-$\delta$ definition to prove that

$$\lim_{x\to 2} \frac{x^2+1}{x} = \frac{5}{2}.$$

Fix $\varepsilon>0$. We want to find $\delta=\delta_\varepsilon>0$ such that $\left|\frac{x^2+1}{x}-\frac{5}{2}\right|<\varepsilon$ if $0<|x-2|<\delta$.

Assume $0<|x-2|<\delta$ where $\delta>0$ is to be chosen to make sure that

$$\left|\frac{x^2+1}{x}-\frac{5}{2}\right|<\varepsilon.$$

We have

$$\frac{x^2+1}{x}-\frac{5}{2} = \frac{2x^2-5x+2}{2x} = \frac{(x-2)(2x-1)}{2x}$$

Thus,

$$\left|\frac{x^2+1}{x}-\frac{5}{2}\right| = \frac{|x-2||2x-1|}{2|x|} \leq \frac{\delta|2x-1|}{2|x|}$$

Assume $\delta<1$. Then $|x-2|<\delta<1$. Then $-1<x-2<1$. Then $1<x<3$.

We have

$$|2x-1| = 2x-1 > 2(3)-1 = 5$$

$$|x| = x < 1$$

Thus,

$$\left|\frac{x^2+1}{x}-\frac{5}{2}\right| \leq \frac{\delta|2x-1|}{2|x|} < \frac{\delta\cdot 5}{2\cdot 1} = \frac{5\delta}{2}$$

For $\left|\frac{x^2+1}{x}-\frac{5}{2}\right|<\varepsilon$, we will ask that $\frac{5\delta}{2}\leq\varepsilon$ and $\delta<1$. This is satisfied

by choosing $\delta=\min\left\{\frac{2\varepsilon}{5},\frac{1}{2}\right\}$.

## * A note about cluster points

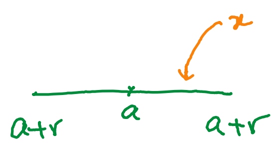
$f(n) = n^2 \quad (n \in \mathbb{N})$

$\lim_{n \to 2} f(n)$ is not meaningful because there are no natural numbers that are close to 2. (2 is an isolated point)

$\lim_{x \to a} f(x)$ is meaningful only if $a$ is a cluster point of the domain of $f$.

$a$ is a cluster point (or limit point, or accumulation point) of a set $D$ if

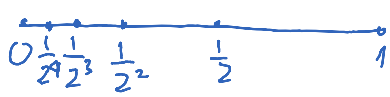
$\forall r > 0,\ \exists\, x \in (D \cap (a-r, a+r)) \setminus \{a\}$.




Note that $a$ doesn't necessarily belong to $D$ to be a cluster point of $D$.

Ex $\sqrt{2}$ is a cluster point of $\mathbb{Q}$ but $\sqrt{2} \notin \mathbb{Q}$.

Ex the sequence $\left\{ \frac{1}{2^n} : n \in \mathbb{N} \right\}$ has only one cluster point, which is 0.




Ex The sequence

$$a_n = \begin{cases} 1 + \frac{1}{n} & \text{if } n \text{ is odd} \\ -1 + \frac{1}{n} & \text{if } n \text{ is even} \end{cases}$$

has two cluster points (1 and $-1$).

Ex The sequence

$$a_n = \begin{cases} p + \frac{1}{k} & \text{if } n = p^k, k \in \mathbb{N}, p \text{ is a prime number} \\ n & \text{otherwise} \end{cases}$$

has infinitely many cluster points. Every prime number is a cluster point of $\{a_n : n \in \mathbb{N}\}$.