Lecture 15

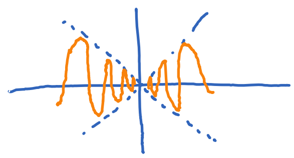
Wednesday, October 9, 2024 1:52 PM

Continuity:

Function $f: D \subset R \to R$ is **continuous** at $a \in D$ if either a is an isolated point of D or a is a cluster point of D and $\lim_{x\to a} f(x) = f(a)$.

Ex: any sequence is a function from N to R. Because every point in N is an isolated point, the function is continuous at every point in N.

Ex: function
$f(x) = x\sin\left(\frac{1}{x}\right)$ if $x \neq 0$
$f(0) = 0$
is continuous at 0. One can use the $\epsilon\delta$ definition of limit to show this. Note that it is impossible to draw the graph of this function from $x = -1$ to $x = 1$ using one stroke of a pencil.



Ex: Thomae's function $f: (0,1) \to R$ defined by
$f(x) = 0$ if $x \in (0,1)\backslash Q$
$f(x) = \frac{1}{q}$ if $x = \frac{p}{q} \in (0,1) \cap Q$, where $\frac{p}{q}$ is a reduced fraction and $q > 0$.
This function is continuous at every irrational number and discontinuous at every rational number.

Let $a = \frac{p}{q} \in (0,1)$ and a is rational. Then $f(a) = \frac{1}{q} > 0$. There exists a sequence of irrational numbers x_n such that $x_n \to a$. Indeed, one can choose $x_n = a + \frac{\sqrt{2}}{n}$
Then $f(x_n) = 0$ and $\lim_{n\to\infty} f(x_n) = 0 \neq f(a)$. Thus, f is discontinuous at a.

Let $a \in (0,1)\backslash Q$. Then $f(a) = 0$. Let (x_n) be any sequence converging to a. If there are finitely many every rational numbers in this sequence, then x_n is irrational for n sufficiently large. Thus, $f(x_n) = 0$. If there are inifinitely many rational numbers in the sequence (x_n), one can split the sequence into an irrational part and a rational part (subsequence).

It suffices to consider the case where $x_n \in Q$ for all n. Write $x_n = \frac{p_n}{q_n}$. As $x_n \to a$ but not equal to a, q_n must go to infinity.