Lecture 17

Monday, October 14, 2024 5:25 AM

In the proof of the fact that every continuous function on $[a, b]$ is bounded which I gave you last time, where is the closed interval $[a, b]$ needed (as opposed to (a, b) or $(a, b]$ or $[a, b)$)?

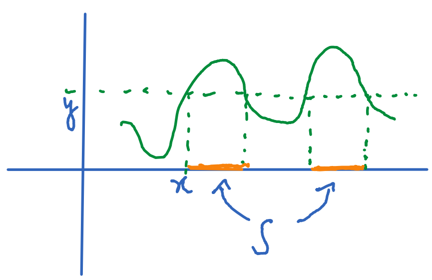
Theorem: every continuous function on $[a, b]$ attains a maximum value and a minimum value on $[a, b]$.

Proof: Let $M = \sup f$. There exists a sequence (x_n) in $[0,1]$ such that $M - \frac{1}{n} < f(x_n) \leq M$ for all n. Thus, the sequence $(f(x_n))$ converges to M. Then use a subsequence argument (B-W theorem).

Theorem: Let f be a continuous function on $[a, b]$ and $\min f < y < \max f$. There exists $x \in [a, b]$ such that $f(x) = y$.

Proof: The minimum value of f is attained at x_1 and the maximum value of f is attained at x_2. Suppose $x_1 < x_2$. Let

$$x = \inf S \text{ where } S = \{z \in (x_1, x_2): f(z) > y\}$$



Note that the set S is nonempty because it contains x_0. Thus, x is finite and $x \in [x_1, x_2]$. We claim that $f(x) \geq y$. Use sequence argument: let (z_n) be a sequence in S such that $(z_n) \to x$. Then $f(z_n) \to f(x)$ thanks to the continuity of f. Because $f(z_n) \geq y$ for all n, we must have $f(x) \geq y$.

If $x = x_1$ then $f(x_1) \geq y$, which means $\min f \geq y$, which is a contradiction.

Therefore, $x > x_1$. For sufficiently large n, we have $x_1 < x - \frac{1}{n} < x$. Thus, $x - \frac{1}{n}$ belongs to (x_1, x_2) but does not belong to S. Thus, $f\left(x - \frac{1}{n}\right) \leq y$. Then

$$f(x) = \lim_{n \to \infty} f\left(x - \frac{1}{n}\right) \leq y$$

Therefore, $f(x) = y$.

In the case $x_1 > x_2$, let

$$x = \sup S \text{ where } S = \{z \in (x_1, x_2): f(z) > y\}$$

And the remaining argument follows the same lines as above.