Lecture 2

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Real number "package" $(\mathbb{R}, +, \cdot, <)$ is <u>defined</u> by axioms:

- addition
  - → closed
  - → commutative
  - → associative
  - → having a neutral element "0"
  - → each element having an additive inverse
- multiplication
  - → closed
  - → commutative
  - → associative
  - → unit element "1"
  - → nonzero element having a multiplicative inverse
- distribution between "+" and "·"
- order
  - → trichotomy
  - → transitive
- $x<y \Rightarrow x+z<y+z$
- $0<x, 0<y \Rightarrow 0<xy$
- least upper bound exists.

From all the above axioms, we can build everything else about real numbers

What is number 2?

$2 = 1+1$

$3 = 1+1+1 = 2+1$

What is $\frac{2}{3}$?

$$\frac{2}{3} = 2 \cdot 3^{-1}$$

So, we can construct the rational numbers as a subset of real numbers.

Ex Show that $x0 = 0$ for all $x \in \mathbb{R}$

Ex Show that $-x = (-1)x$ for all $x \in \mathbb{R}$

Ex Show that $(-1)(-1) = 1$

Ex Show that $(-x)(-x) = x^2$ for all $x \in \mathbb{R}$

$x > y$ means $y < x$

$x \leq y$ means $x < y$ or $x = y$

$x \geq y$ means $x > y$ or $x = y$

Ex Let $x > y$. Show that $x - y > 0$.

* Supremum and infimum

$A \subset \mathbb{R}$ is a nonempty subset.

$\sup A =$ the smallest upper bound of A

$\inf A =$ the greatest lower bound of A

To show that $\sup A = m$, we need to show that

$$\begin{cases} m \text{ is an upper bound of } A \\ \text{If } M \text{ is an upper bound of } A \text{ then } M \geq m. \end{cases}$$

Ex Let $A = [1, 2)$. Find $\sup A$ and $\inf A$

Ex Let $A = \left\{ \frac{n}{n+1},\ n = 1, 2, 3, \dots \right\}$

Find $\sup A$ and $\inf A$.

Ex Let f, g be bounded functions from $[0,1]$ to $\mathbb{R}$.

$$A = \sup\{f(x) \mid x \in [0,1]\}$$

$$B = \sup\{g(x) \mid x \in [0,1]\}$$

$$C = \sup\{f(x) + g(x) \mid x \in [0,1]\}$$

Show that $\sup C \le \sup A + \sup B$

$\inf C \geq \inf A + \inf B$

Give an example where the equality doesn't hold.

* Archimedean property:

If $x > 0$ then there exists $n \in \mathbb{N}$ such that $nx > 1$.

Proof by contraction: suppose that $nx \le 1$ for every $n \in \mathbb{N}$.

Then $n \le \frac{1}{x}$ for every $n \in \mathbb{N}$

That means $\frac{1}{x}$ is an upper bound of $\mathbb{N}$. Thus, $\mathbb{N}$ has a supremum, called M.

Because $M - 1 < M$, it is not an upper bound of $\mathbb{N}$.

There exists $m \in \mathbb{N}$ such that $m > M - 1$. Then $\underbrace{m+1}_{\in \mathbb{N}} > M$. Then M

is not an upper bound of $\mathbb{N}$. This is a contradiction